Journal of Statistical Physics, Vol 44, Nos. 3/4, 1986

A Heuristic Theory of the Spin Glass Phase

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Received January 20, 1986; final March 20, 1986

We study the low-temperature phase of the nearest-neighbor Ising spin glass. Our analysis of gauge-invariant ground state Peierls contours suggests the existence of infinitely many disjoint Gibbs states at low temperatures, provided the dimension, d, is sufficiently large (presumably $d > 3$ or 4), while for $d = 2$ the Gibbs state is unique for all temperatures. In $d \geqslant 3$ we present arguments supporting the existence of a massless phase with broken spin-flip symmetry at low temperatures.

KEY WORDS: Edwards–Anderson model; low-temperature phase; frustration; percolation; ground state structure; Gibbs states; ultrametricity.

1. INTRODUCTION

Spin glasses are among the least understood systems in equilibrium statistical mechanics. In particular, their low-temperature regime and critical behavior are to a large extent unknown; even the very existence of a spin glass phase transition in three and more dimensions is under dispute.[1–3]

This lack of understanding is due to the fact that not only do we not know of any adequate analytical methods to cope with the spin glass problem, but even numerical (Monte Carlo, etc.) studies meet with formidable difficulties: The dynamical stochastic processes set up to simulate equilibrium states fail to actually reach equilibrium in available computer times if the temperature is low. Hence, so far, all computer studies, even when performed on optimized special purpose computers, were confined to rather high temperatures.[4,5]

Analytical investigations have focused attention on the mean field model of Sherington and Kirkpatrick.[6] Using the replica trick and the concept of replica symmetry breaking, Parisi *et al.*[7,8] have obtained a

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0022-4715/86/0800-0347$05.00/0

rather appealing picture of the low-temperature phase of this model: There exist infinitely many extremal Gibbs states at very low temperature, forming a space with an ultrametric topology, which may be arranged in a generation tree. As the temperature is raised, states within increasing distance from each other coalesce until above a certain freezing temperature the equilibrium state is unique.

Although this picture is rather nice, its derivation suffers from some shortcomings. First, the Sherrington–Kirkpatrick model with its infinite range interaction is somewhat unphysical. In particular, being really a model in infinitely many dimensions, it cannot reflect the dependence of various properties of spin glasses on dimension. Second, the use of the replica trick in this context is disputable. The replica symmetry breaking scheme of Parisi provides a computational method that gives physically sensible results, but its mathematical status is still mysterious.

Given the wide range of phenomena which have been described by spin glass models during the last ten years, reaching beyond the magnetic alloys they were invented for[9] to studies of questions in biology and the theory of memory and computer science,[10] the poor theoretical situation is quite unsatisfactory, and new approaches to the problem seem desirable.

In this paper we propose to study the short-range Ising spin glass (*Edwards–Anderson model*[11]) in zero magnetic field using a geometric, gauge-invariant formulation. The basic objects in this formulation are the distribution of frustration,[12] which is the essential element of disorder in the system, and the *Peierls contours* that describe the distribution of the energy in a given configuration.

Specifically, we study the Hamiltonian

$$H_J = \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \tag{1.1}$$

where σ_i are Ising spins, J_{ij} are independent random variables, and $\langle ij \rangle$ are nearest-neighbor bonds on a lattice $\mathscr{Z}^d$.

In this paper we propose theoretical arguments supporting the following conjectures:

(i) In dimension less than three there is a unique Gibbs state at all temperatures and presumably no phase transition at finite temperature.

(ii) In three dimensions, below some temperature T_0, a spin glass phase with divergent correlation length and broken spin-flip symmetry appears. (However, for all temperatures $T > 0$, there exist at most two extremal Gibbs states related to each other by a global spin flip.)

(iii) Above some critical dimension d (presumably equal to three), there exist infinitely many "*dominant*" ground states separated by infinite energy barriers, which at low enough temperature, give rise to infinitely many disjoint extremal Gibbs states.

The question of the existence of many extremal equilibrium states will be seen to be tied to a question concerning interface fluctuations in a random environment. A precise formulation will be given later, but we wish to note that interfaces and *Bloch* walls in an Ising spin glass have much larger fluctuations than in the Ising ferromagnet and hence tend to be *delocalized* in three dimensions. We expect their *Hausdorff dimension* to be larger than $d-1$.

This paper is organized as follows. In the next section, we present the gauge-invariant formulation of the model (1.1) and discuss the ensuing geometrical structure. In Sec. 3 we analyze the distribution of frustration using some ideas from percolation theory. These results will provide a useful basis for the analysis of the ground state structure that we present in subsequent sections. Some of the results in Sec. 3 represent joint work with Michael Aizenman. In Sec. 4 we discuss some general features of ground states in our models and their relation to Gibbs states at low temperature. In Sec. 5 we study the low-energy excitations above an arbitrary groundstate. We estimate the density of low-energy excitations as a function of energy and argue that in three or more dimensions there exist excitations that are capable of producing long-range correlations. In Sec. 6 we investigate the possible existence of many disjoint Gibbs states. We present arguments that in two dimensions the Gibbs state is unique, for all temperatures, whereas in sufficiently high dimensions one should expect that there are infinitely many disjoint Gibbs states at low enough temperature. In Sec. 7, we summarize our main conclusions, discuss some important thresholds in the spin glass problem, relate them to percolation thresholds, and finally describe some open problems. A sketch of our arguments has appeared in Ref. 13.

2. GAUGE INVARIANT FORMULATION OF A SPIN GLASS MODEL

We consider a spin glass with Hamiltonian[2]

$$H_J = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j \tag{2.1}$$

[2] The external magnetic field is zero.

where σ_i takes the values ± 1. The J_{ij} are independent random variables whose distribution we choose as follows:

$$d\rho(J) = (xg(-J) + (1-x)\, g(J))\, dJ \tag{2.2}$$

where g is some nonnegative function with support on the positive reals, and

$$\int dJg(J) = 1 \tag{2.3}$$

A fairly common choice for g is

$$g(J) = \delta(J_0 - J) \tag{2.4}$$

The singular distribution (2.4) may create some special features like residual entropy at $T=0$,[14] but otherwise the explicit form of the distribution of J is expected to be rather unimportant.

By (2.2), the sign of J is distributed according to a Bernoulli bond percolation process with density x. This fact will be used frequently.

We notice that our model has some gauge invariance. Consider the transformations

$$\sigma_i \to \sigma_i' = \varepsilon_i \sigma_i \tag{2.5}$$

$$J_{ij} \to J_{ij}' = \varepsilon_i \varepsilon_j J_{ij} \tag{2.6}$$

where $\varepsilon_i \in \{+1, -1\}$. Clearly

$$H_{J'}(\sigma') = H_J(\sigma) \tag{2.7}$$

Thus spin glasses with gauge-equivalent configurations, J and J', of exchange couplings [see (2.6)] describe the same magnetic system, as long as the external field vanishes.

An immediate consequence of this gauge invariance is the vanishing of the averaged magnetization when x equals 1/2.[14] Namely, in this case the distribution of J_{ij}, (2.2), is also invariant under the transformation (2.6), and therefore we have

$$\begin{aligned} m &= \int \prod_{\langle ij\rangle} d\rho(J_{ij}) \langle \sigma_0 \rangle_J = \int \prod_{\langle ij\rangle} d\rho(J_{ij}) \langle \varepsilon_0 \sigma_0 \rangle_{J'} \\ &= \varepsilon_0 \int \prod_{\langle ij\rangle} d\rho(J_{ij}') \langle \sigma_0 \rangle_{J'} = \varepsilon_0 m \end{aligned} \tag{2.8}$$

Choosing $\varepsilon_0 = -1$ (and all the others $+1$), this yields $m = -m$, i.e., $m = 0$.

One may ask to what extent the randomness of the signs of the couplings J_{ij} can be gauged away by gauge transformations.

Clearly, a complete removal would only be possible if every J_{ij} were of the form $J_i J_j$. (This is the case in the Mattis model.[15]) To characterize the deviation of a configuration J_{ij} from such a trivial one, it is useful to introduce the concept of "*frustration*."[12] Let us denote by p a plaquette (or elementary two cell) of the lattice. Define

$$\tau_p = \prod_{\langle ij \rangle \in \partial p} \operatorname{sign}(J_{ij}) \tag{2.9}$$

We say that p is "*frustrated*" whenever $\tau_p = -1$. One easily verifies that τ_p is gauge invariant and that, for trivial configurations, τ_p is positive on all plaquettes. Furthermore, gauge *inequivalent* configurations of J_{ij}'s give rise to *different configurations* of frustrated plaquettes.

It is convenient to associate frustration with cells in the dual lattice, i.e., in two dimensions we associate frustration with the sites dual to the frustrated plaquettes, in three dimensions with the dual bonds, and in general with the dual $(d-2)$-cells. We denote the resulting sets dual to frustrated plaquettes by Φ.

In three dimensions, the set Φ consists of a collection of (possibly unbounded) loops. This important property is easily understood if we consider an elementary cube c in the direct lattice. If a line in Φ were to end in this cube, an odd number of them would have to enter it, and thus the quantity

$$\prod_{p \in \partial c} \tau_p$$

would have to be negative. But

$$\prod_{p \in \partial c} \tau_p = \prod_{\langle ij \rangle \in \partial c} (\operatorname{sign}(J_{ij}))^2 \equiv +1 \tag{2.10}$$

which proves that no frustration line may end in any cube; hence it must form a loop. Relation (2.10) is (in differential geometric language) known as the *Bianchi identity*.

In general, (2.10) shows that no complex made of $(d-2)$-cells dual to frustrated plaquettes may end in any cube. Thus, in general, frustrated plaquettes are dual to closed complexes of $(d-2)$-cells.

A further ingredient for a complete gauge-invariant description of our model is the notion of gauge-invariant Peierls contours.

Connected components of the set of cells $\langle ij \rangle^*$ dual to bonds $\langle ij \rangle$ with $\operatorname{sign}(J_{ij}\sigma_i\sigma_j) = -1$ are called Peierls contours and are denoted by γ.

Furthermore, Γ denotes the set of all Peierls contours in a given configuration of spins.

In unfrustrated systems Peierls contours form a collection of loops (resp. closed surfaces, $d-1$-complexes). In the presence of frustration this is no longer the case. By the very definition of frustration, an odd number of bonds of a frustrated plaquette must be energetically unfavorable, i.e., must be dual to a cell of a Peierls contour. Therefore the frustration network Φ forms the boundary of the Peierls contours, i.e. (see also Fig. 1),

$$\partial\Gamma = \Phi \tag{2.11}$$

(Note that we are considering an infinite system at the moment. Appropriate modifications for finite systems with boundaries will be discussed is Sec. 4.)

We may now express the energy of a configuration of spins entirely in terms of gauge-invariant quantities, namely,

$$E(\Gamma) = 2 \sum_{\langle ij\rangle^* \in \Gamma} |J_{ij}| \tag{2.12}$$

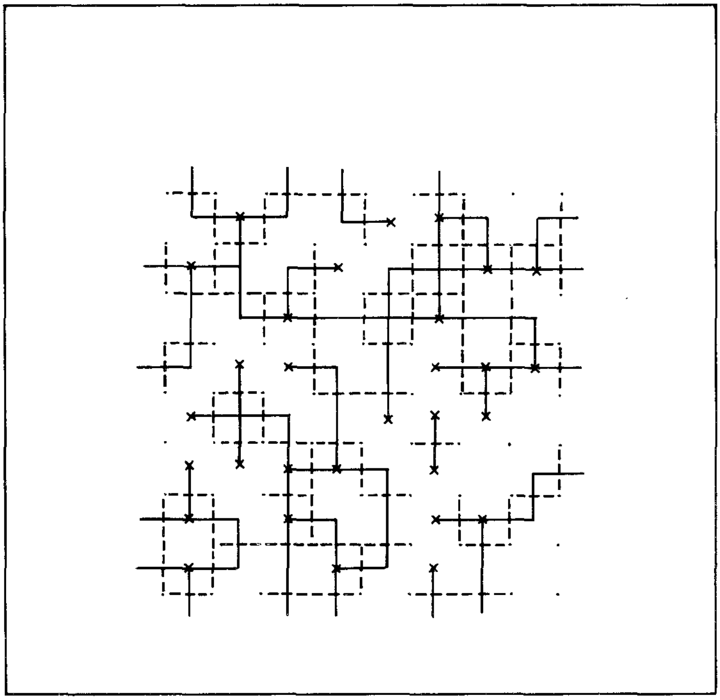

Fig. 1. Sites dual to frustrated plaquettes (crosses), bonds with negative J_{ij} (dotted lines), and Peierls contours (solid lines) in a configuration with all spins up.

(This is the value the Hamiltonian takes on a spin configuration whose Peierls contours are given by Γ, up to an overall constant.)

We conclude that a complete gauge-invariant description of the spin glass involves specifying

(i) the frustration network Φ

(ii) the moduli of the couplings, $|J_{ij}|$

(iii) a global sign of the spin configuration

It is useful to exhibit also the gauge degrees of freedom in geometrical terms. Consider contours, B, that consist of cells dual to bonds $\langle ij \rangle$ for which J_{ij} is negative. Again, a frustrated plaquette has by definition an odd number of such bonds in its boundary, and thus the cell dual to it is in the boundary of an odd number of elementary cells of a B-contour. Therefore again

$$\partial B = \Phi \tag{2.13}$$

A gauge transformation (2.6) acts on these B-contours as a deformation that leaves their boundary invariant.

The B-contours give us another way of visualizing the frustration network. The cells dual to negative bonds are simply distributed according to a Bernoulli $(d-1)$-cell percolation process. By (2.13), the frustration network Φ is nothing but the boundary (mod 2) of such a cell complex. The fact that in three dimensions Φ is made of loops is a direct consequence of this observation.

Finally we may consider another type of contours, namely those across which spins are flipped, i.e., the Bloch walls. Bloch walls are, however, easily expressed in terms of the Γ and B contours. For, if a spin flip occurs across some dual cell, then either this cell belongs to a Γ-contour and not to a B-contour, or it belongs to a B-contour and not to a Γ-contour. Thus the Bloch walls, W, are just the symmetric difference of the Γ- and B-contours,

$$W = \Gamma \Delta B \tag{2.14}$$

(The symmetric difference of two sets A, B is defined as $A \Delta B \equiv (A \cup B) \backslash (A \cap B)$.)

Since $\partial \Gamma = \partial B = \Phi$, we see that

$$\partial W = \varnothing \tag{2.15}$$

as must be.

Clearly knowledge of the Bloch walls allows one to reconstruct the configuration of the spins $\{\sigma_i\}$ up to a global spin flip. By (2.14), this can

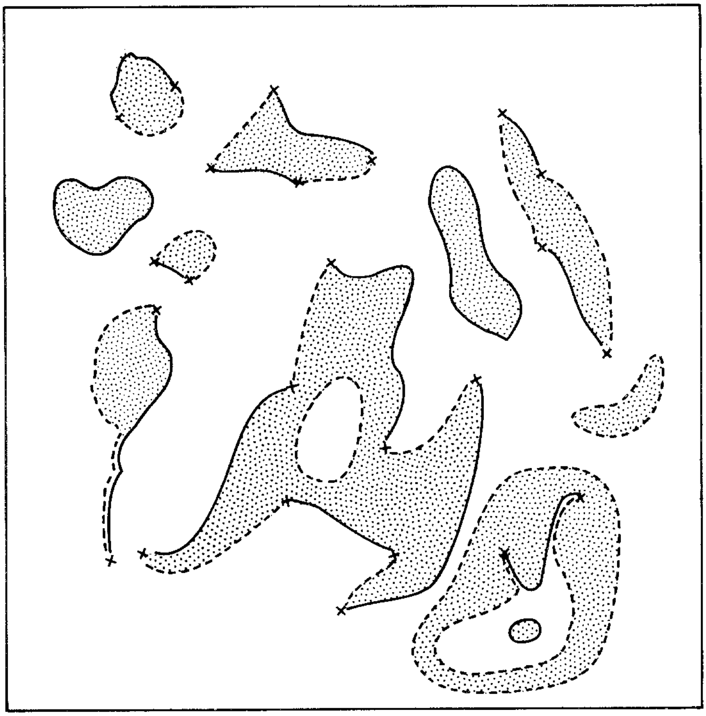

Fig. 2. Sites dual to frustrated plaquettes (crosses), Peierls contours (solid lines), and contours dual to bonds with negative J (dotted lines). The shaded regions have spins down, the blank ones spin up.

be done if we know the Peierls contours and the J_{ij} configuration (i.e., the B-contours). An arrangement of frustration, Γ- and B-contours, and the corresponding spin configurations in two dimensions is shown in Fig. 2.

To illustrate once more the connection between the spins and the Peierls contours, let us consider what happens if we flip all the spins within a region, D, say. Clearly, contours can change only on the boundary of the dual of D, ∂D^*. If an element of this boundary had been an element of a Peierls contour before the flip, it will no longer be one afterwards, and vice versa; i.e., a spin flip within D interchanges Peierls contours with non-Peierls contours on ∂D^*.

3. PERCOLATION OF FRUSTATION

We have seen in the last section that the negative sign of the couplings J_{ij} are distributed according to a Bernoulli bond percolation process[16] with density x. From this fact it is possible to infer a considerable amount of information on the distribution of frustation. This is done in the present section.

Since the nature of "*frustration networks*" is quite different in two and in higher dimensions, we treat these cases essentially separately. The main results of this section are described in the following two lemmas.

Lemma 3.1. In two dimensions we have:

(i) If $x = 1/2$, frustrated plaquettes are independently distributed with density 1/2.

(ii) For x near 1/2, frustrated plaquettes star-percolate, i.e., there exists an infinite, star-connected cluster of frustrated plaquettes, two plaquettes being considered connected if they have at least one point in common.

(iii) For all x, the probability that a frustrated plaquette is far away from the next one is very small. Specifically, let $e(D)$ be the event that a frustrated plaquette is surrounded by a disk D free of frustration. Then

$$[2x(1-x)]^{|D|-5}c(x) \leqslant \text{Prob}\{e(D)\} \leqslant [x^2+(1-x)^2]^{|D|-5}c(x) \quad (3.1)$$

where

$$c(x) = \frac{1}{x^2+(1-x)^2}\{(1-x)^2[(3x(1-x)^2+x^3)(3x^2(1-x)+(1-x)^3)^3] + x^2[(3x(1-x)^2+x^3)^3(3x^2(1-x)+(1-x)^3)]\} \quad (3.2)$$

Furthermore, let $r(D)$ be the length of the shortest path connecting the frustrated plaquette with the exterior of D. Then, for fixed $r(D)$, we have

$$\text{Prob}\{e(D)\} \sim x^{r(D)}(1+O(x)) \qquad \{\text{resp. } (1-x)^{r(D)}(1+O(1-x))\} \quad (3.3)$$

asymptotically as $x \to 0$ {resp. $x \to 1$}.

Lemma 3.2. In dimensions d greater or equal to three:

(i) If $x = 1/2$, frustrated plaquettes are "almost" independently distributed, that is, the only correlations are due to the constraint that the $(d-2)$-cells dual to frustrated plaquettes form closed complexes.

(ii) For x "near" 1/2, there exists a unique, infinite connected cluster, Φ_∞, of $(d-2)$-cells dual to frustrated plaquettes.[3]

A remark on what is meant by "near 1/2" in these statements is appropriate. The results claimed are most easily obtained for $x = 1/2$. To be sure they are relevant for the spin glass (whose behavior should not depend

[3] This result was first obtained by Michael Aizenman.

dramatically on x being exactly equal to 1/2), we must extend them to a neighborhood of 1/2. In order to have simple and transparent proofs, we do not endeavor getting optimal results on the range of x. We will, however, occasionally give some numerical values for what is to be considered "near" in reality.

Let us now show how such results are derived.

First of all, one may calculate the average density of frustrated plaquettes, i.e., the probability that a given plaquette, p, is frustrated

$$\mathrm{Prob}\{\tau_p = -1\} = \frac{1}{2} - 8\left(x - \frac{1}{2}\right)^4 \tag{3.4}$$

Unfortunately, this is not enough information to calculate the probabilities of more complicated events, since frustrated plaquettes are not independently distributed.

To overcome this difficulty, we want to derive bounds for expectation values in terms of expectations in a simple Bernoulli ensemble of independently distributed plaquettes. More precisely, we will show that there is a Bernoulli process with some density (depending on x and equal to 1/2 if $x = 1/2$) such that expectations of positive events in this ensemble are uniform lower bounds for the expectations of the corresponding events in the ensemble of frustrated plaquettes.

Let A_n be a set of n plaquettes. We propose to estimate the probability that all the plaquettes of A_n are frustrated (the plaquettes not belonging to A_n may or may not be frustrated). Call this event α_n. We define inductively certain classes of sets A_n. Let $\mathscr{A}_1^k$ be all sets consisting of a single plaquette. We say that $A_n \in \mathscr{A}_n^k$ if by cutting no more than k edges a plaquette p can be removed from A_n, and the resulting set is contained in $\mathscr{A}_{n-1}^k$. (Clearly, in two dimensions all sets of n plaquettes belong to $\mathscr{A}_n^2$.)

We now prove the following bounds:

If $A_n \in \mathscr{A}_n^1$, then

$$\mathrm{Prob}\{\alpha_n\} \geqslant [4x^3(1-x) + 4(1-x)^3 x]^n \tag{3.5}$$

If $A_n \in \mathscr{A}_n^2$, then

$$\mathrm{Prob}\{\alpha_n\} \geqslant [2x(1-x)]^n \tag{3.6}$$

If $A_n \in \mathscr{A}_n^3$, then

$$\mathrm{Prob}\{\alpha_n\} \geqslant x^n \tag{3.7}$$

[(3.7) holds for $x < 1/2$. For $x > 1/2$, the bound is $(1-x)^n$.]

For $x = 1/2$ we get in all three cases the equality

$$\text{Prob}\{\alpha_n\} = (1/2)^n \tag{3.8}$$

(Note that in three and more dimensions there exist sets that are not contained in any one of the above classes.)

All these estimates are easily proven by induction. Let us exemplify this for the case (3.6). Suppose it holds for $n-1$. By definition we can decompose any set $A_n \in \mathscr{A}_n^2$ into a plaquette p and a set A_{n-1} such that p and A_{n-1} share no more than two bonds. We have thus

$$\text{Prob}\{\alpha_n\} = \int \prod_{\langle ij \rangle \not\subset \partial p} d\rho(J_{ij}) \int \prod_{i=1}^{4} d\rho(J_i)\, \chi[\alpha_{n-1}]\, \chi[\tau_p = -1] \tag{3.9}$$

where J_1 and J_2 are the bonds shared by p and A_{n-1}, while J_3 and J_4 are the remaining bonds of p. Distinguishing the possible cases, we get

$$\begin{aligned}
\text{Prob}\{\alpha_n\} = &\int \prod_{\langle ij \rangle \not\subset \partial p} d\rho(J_{ij}) \int dJ_1\, dJ_2 \chi[\alpha_{n-1}] \\
&\times [g(-J_1)\, g(-J_2)\, 2x^3(1-x) + g(J_1)\, g(J_2)\, 2x(1-x)^3 \\
&+ (g(-J_1)\, g(J_2) + g(J_1)\, g(-J_2))(x(1-x)^3 + x^3(1-x))] \\
= &\int \prod_{\langle ij \rangle} d\rho(J_{ij})\, \chi[\alpha_{n-1}]\, 2x(1-x) \\
&+ \int \prod_{\langle ij \rangle \not\subset \partial p} d\rho(J_{ij}) \int dJ_1\, dJ_2 \chi[\alpha_{n-1}] \\
&\times 4\left(x - \frac{1}{2}\right)^2 x(1-x)(g(-J_1)\, g(J_2) + g(J_1)\, g(-J_2))
\end{aligned} \tag{3.10}$$

Since α_{n-1} is independent of J_1 and J_2, the first term in the sum is simply $2x(1-x)$ Prob$\{\alpha_{n-1}\}$. The second is postive and vanishes for $x = 1/2$. This proves (3.6) and (3.8) for $k = 2$. The other relations are proven in the same manner.

Quantitatively, the error commited in throwing away the second term in (3.10) can be fairly large. In fact, with some extra work it should be possible to prove that (3.5) holds in all three cases.

Part (i) of Lemma 3.1 follows now directly from (3.8). Part (ii) follows from the fact that in two dimensions all sets of plaquettes are in class $\mathscr{A}_n^3$. Thus we have the uniform lower bounds (3.7) corresponding to ordinary percolation. Since it is known that two-dimensional site percolation processes star-percolate above a critical density p_c strictly less than 1/2, frustration star-percolates certainly if $2x(1-x) > p_c$.

To prove part (iii), let us first consider the event that the four neighbors $p_1,..., p_4$ of a plaquette p are all unfrustrated, given that p is frustrated. One easily finds, just counting possibilities,

$$\begin{aligned}
&\mathrm{Prob}\{\tau_{p_i}=+1, i=1,..., 4 \mid \tau_p=-1\}\\
&\quad=\frac{1}{x^2+(1-x)^2}\\
&\qquad\times\{(1-x)^2[(3x(1-x)^2+x^3)(3x^2(1-x)+(1-x)^3)^3]\\
&\qquad+x^2[(3x(1-x)^2+x^3)^3(3x^2(1-x)+(1-x)^3)]\}\equiv c(x)
\end{aligned}\tag{3.11}$$

Note that this is a small number for all possible values of x.

Equation (3.1) is then obtained in the same fashion as Eq. (3.6), reducing the disk D, one plaquatte after another, using the inequality

$$2x(1-x)\,\mathrm{Prob}\{e(D\backslash p)\}\leqslant \mathrm{Prob}\{e(D)\}\leqslant[(x^2+(1-x)^2]\,\mathrm{Prob}\{e(D\backslash p)\}\tag{3.12}$$

which is obtained through a calculation analogous to Eq. (3.10), until we are left with just four plaquettes adjacent to the frustrated one, for which we then use (3.11).

The asymptotic behavior for x small [resp. $(1-x)$ small], (3.3), follows from the fact that in order to have no frustrated plaquette a distance r away from a given one requires the existence of at least r negative [resp. positive] bond variables J_{ij}.[4] Thus the leading term in a power-series expansion of $\mathrm{Prob}\{e(D)\}$ is of the form const $\times\, x^{r(D)}$ [resp. $(1-x)^{r(D)}$].

Let us now turn to three or more dimensions. In this case there clearly are sets of plaquettes that are in none of the classes $\mathscr{A}_n^k$, for $k=1, 2, 3$. For events in $\mathscr{A}_n^4$ there is no immediate bound like (3.5)–(3.8). However, any set not in $\mathscr{A}_n^3$ must contain a closed surface. We may now reduce a set A_n not in $\mathscr{A}_n^3$, as outlined in the proof above, until we reach a closed surface, $\mathscr{S}$, that we cannot further reduce in this way. Let $|\mathscr{S}|$ be equal to $2k$. (Obviously, the number of plaquettes in a closed surface is even.) Suppose $2k-1$ of these plaquettes are frustrated. Then due to the loop constraints, the last plaquette is necessarily frustrated, as well. Thus if we remove any plaquette from $\mathscr{S}$, we have

$$\mathrm{Prob}\{\mathscr{S}\}=\mathrm{Prob}\{\mathscr{S}\backslash p\}>\mathrm{Prob}\{\mathscr{S}\backslash p\}\,x\tag{3.13}$$

Using this fact, we may establish (3.7) for all possible sets of n plaquettes in three dimensions. Note, however, that, since (3.13) involves a strict

[4] Here and throughout this paper all distances are measured in units of the lattice spacing.

inequality even if $x=1/2$, (3.8) does not hold in general in dimensions greater than two. This, of course, reflects the existence of the nontrivial constraints that cells dual to frustrated plaquettes must form closed networks.

We now turn to the proof of Lemma 3.2. We specialize to three dimensions for easier visualization, and also because it is the most difficult case. The strategy in higher dimensions is exactly the same.

Consider a two-dimensional sublattice consisting of two adjacent lattice planes:

$$\mathscr{L}^a = \{ i \in \mathscr{Z}^{3*} \mid i = a + ne_x^* + me_y^* + \varepsilon e_z^*,\ m, n \in \mathscr{Z}, \varepsilon \in \{0, 1\}\} \tag{3.14}$$

Clearly, any set A_n dual to a set $A_n^* \subset \mathscr{L}^a$ belongs to $\mathscr{A}_n^3$. Thus the bonds in $\mathscr{L}^a$ dual to frustrated plaquettes are independently distributed for $x=1/2$, while in general they satisfy (3.7).

Now note that $p=1/2$ is the threshold for bond percolation in two dimensions. Percolation in a double layer is more likely and occurs above some critical density that is strictly lower than 0.45. To see why this is true, consider the two lattice planes L_1 and L_2 that make up $\mathscr{L}^a$. We may define effective bond weights, f_{ij}, on the bonds in L_1, by setting $f_{ij} = -1$ if either the bond $\langle ij \rangle$ is dual to a frustrated plaquette, or if the three bonds in $\mathscr{L}^a$ forming a "bridge" over $\langle ij \rangle$ are dual to frustrated plaquettes. Obviously, if the density of frustrated plaquettes is x, then the density of bonds with negative weights is equal to $x + x^3(1-x)$. The negative f^{ij} are not independently distributed (not even if $x=1/2$), but using the results above and similar arguments one may show that expectations of positive events are bounded by expectations in a Bernoulli bond percolation ensemble with density $\xi = x + (1-x)\,x^3$ [for $x<1/2$; for $x>1/2$ interchange x and $1-x$.] If $0.55 > x > 0.45$, this density is above 1/2 and bonds with negative f_{ij} percolate.[(16)] But then the bonds dual to frustrated plaquettes in $\mathscr{L}^a$ must also percolate.[5]

The double layers $\mathscr{L}^a$ may be used as building blocks for more complicated events. The important feature for this to work is that if in two intersecting double layers clusters of negative weights intersect, then the same is true for the real frustration.

The first result we want to derive from the existence of percolation in double layers is the existence of a unique, infinite cluster of frustration if $0.55 > x > 0.45$ [see remark above]. The existence of an infinite cluster is already proven, since there exists one even in each double layer. We are left to show the uniqueness. For this, suppose there are two pieces of the infinite clusters entering a cubic box of side length L but that do *not* inter-

[5] This estimate for the percolation threshold of $x=0.45$ is clearly too high. The actual value is more likely to be around 0.23.

sect inside it. We claim that, with probability one, they are connected outside of the box. To show this we place six double layers parallel to the six sides of the box and a distance R away from its center.

Clearly, each of the two pieces of infinite clusters has to cross at least one of the double layers. If we show that with sufficiently large probability these intersection points are connected within these double layers, then we are done. To show that this is in fact the case, it suffices to use again our bounds in terms of Bernoulli percolation with density x. As we have seen before, in such an ensemble there is an infinite cluster with a positive density ξ_0 in each double layer. Thus with probability $>\xi_0^2$ both pieces intersect one of these infinite clusters. Furthermore, with a probability larger than ξ_0^6 the six infinite clusters in the double layers form one connected network. (Note that we are dealing with a Bernoulli ensemble in order to derive our bounds.) Hence a connection between the two pieces is achieved with probability at least ξ_0^8, within these six double layers, independent of their distance R from the cube's center. Since there are infinitely many such sets of double layers, the probability that a connection is achieved in some of them is, by the *Borel–Cantelli lemma*, equal to one. This finishes the proof of Lemma 3.2.

It seems worthwhile to stress at this point that the study of frustration networks constitutes in itself an interesting problem of percolation theory that has, to our knowledge, not received much attention so far.

An interesting feature of the problem is that it may be studied from two quite different points of view. One way to look at it is to consider the frustrated plaquettes, which are correlated random variables, as the basic objects and to study their distribution. The other is to start from the cells dual to negative J_{ij}'s, which are independently distributed, and investigate the structure of their $\mathscr{Z}_2$-boundary. The percolation threshold for this boundary apparently characterizes a new critical point in the bond percolation model.

We have done some preliminary and rather pedestrian numerical simulations on this model on an IBM PC AT. We were working with a maximal lattice size of $30\times 30\times 30$. Our results indicate that the critical density x_b for the appearance of an infinite frustration network is

$$x_b = 0.081 \pm 0.005 \qquad (\text{resp. } 0.919) \tag{3.15}$$

Furthermore, the density of the infinite frustration network apparently goes to zero at this point continuously with a power law

$$\rho_\infty \sim (x - x_b)^{\beta_F} \tag{3.16}$$

where

$$\beta_F \approx 0.3 \pm 0.1 \tag{3.17}$$

This probability can be made quite close to one by choosing λ somewhat large, but only of the order of $\ln L$. Such sharply localized events will be used in Section 6 to argue that there are long-range correlations at low temperature.

4. THE GENERAL STRUCTURE OF GROUND STATES

The low-temperature regime of a spin glass is governed, as we argue, by the peculiarities of the structure of its ground states. In the present section we provide the conceptual armament to exhibit these ground state properties and to relate them to finite temperature quantities. We feel that such conceptual clarification is useful and necessary in view of the considerable confusion that permeates the spin glass literature regarding these questions. We will follow to some extent Ref. 17 and use the infinite system formalism.

As in the previous chapters, Γ denotes a collection of Peierls contours $\gamma_1,..., \gamma_n$, in $\mathscr{L}^d$, which are either closed, or whose boundary is in the frustration network Φ. More precisely, we have $\partial\Gamma = \Phi$. Recall that, given the configuration of couplings, $\{J_{ij}\}$, Γ determines the configuration of spins, but *only* up to a *global spin-flip*. Thus, although we will call Γ a "*configuration*" sometimes, we keep in mind that it corresponds to two different spin configurations, (Γ, σ), with $\sigma \in \{+, -\}$ denoting the global spin orientation.

To define thermodynamic quantities, we have to consider restrictions of our system to finite volumes, Λ. We denote the restriction of Φ to Λ by Φ_Λ, and the restriction of Γ to Λ by Γ_Λ. Note that whereas for the infinite system $\partial\Gamma = \Phi$, the finite volume restrictions satisfy

$$\partial\Gamma_\Lambda = \Phi_\Lambda \cup (\Gamma \cap \partial\Lambda) \tag{4.1}$$

The set $\Gamma \cap \partial\Lambda$ specifies the *boundary conditions* (b.c.) Γ imposes on the box Λ, again up to the global spin-flip specification. In two dimensions, it consists of a set of sites which are to be considered as "frustrated." The b.c. are always such that the number of sites dual to frustrated plaquettes inside Λ plus the number of these additional frustrated sites in $\partial\Lambda$ is even. In three dimensions, the b.c. are lines in $\partial\Lambda$, such that $\Phi_\Lambda \cup (\Gamma \cap \partial\Lambda)$ is a collection of loops, etc.

Let us now define a set, G_Λ, of configurations that minimize the energy locally: We say that $\Gamma \in G_\Lambda$, if and only if for all Γ' such that $\Gamma'_{\Lambda^c} = \Gamma_{\Lambda^c}$,

$$E(\Gamma'_\Lambda) \geqslant E(\Gamma_\Lambda) \tag{4.2}$$

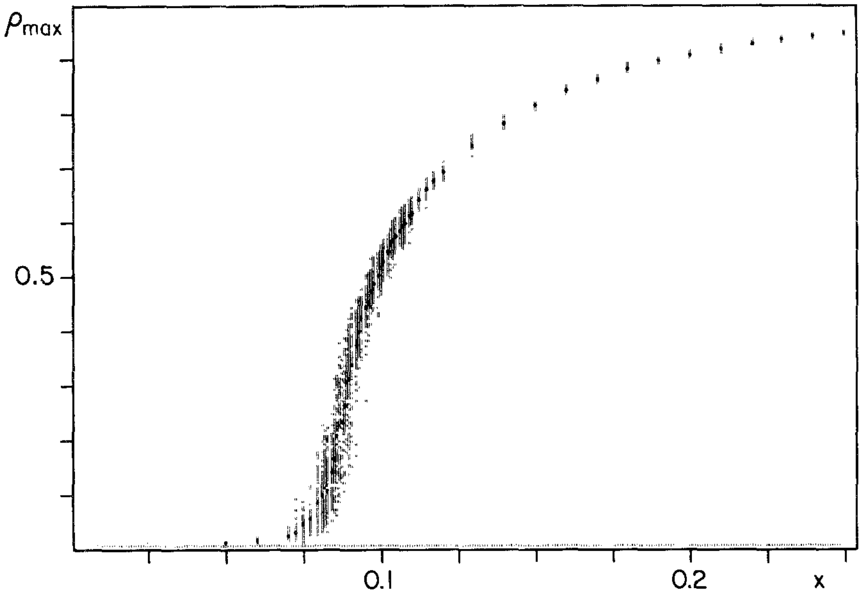

Fig. 3. Density of the largest connected cluster of bonds dual to frustrated plaquettes as a function of x.

Our measured data are presented in Fig. 3. Obviously, these results are very qualitative, and we do not claim high accuracy or reliability. A more serious numerical study of this system would be rather useful.

Before closing this section we will give another example of how double layers may be used to construct interesting events. Consider a cylinder made of four rectangular pieces of double layers, of size $L \times \lambda$, the sides of length λ being glued together. We denote by $C(L, \lambda)$ the event that a frustration loop winds around this cylinder within these double layers. Clearly this event occurs if a loop of bonds with negative effective weights f_{ij} winds around it. Suppose this does not happen. Then, there must be a path of dual bonds crossing the cylinder from the left to the right, such that none of these bonds crosses an "effectively frustrated" bond. But it is well known that if the direct bonds percolate, then the probability that two points are connected by a path as described decays exponentially with their distance.[16] The probability of a left-right crossing of the cylinder is thus easily bounded from above by

$$\frac{4L}{m} e^{-m\lambda} \tag{3.18}$$

where $m > 0$. Thus

$$\text{Prob}\{C(L, \lambda)\} > 1 - \frac{4Le^{-m\lambda}}{m} \tag{3.19}$$

Note that, if Λ is a bounded subset of $\mathscr{Z}^d$, $E(\Gamma_\Lambda)$ is finite and well defined for all Γ.

A "*ground state contour*" is now defined as a contour, Γ, which belongs to G_Λ for all finite boxes Λ. Thus the set, $\mathscr{G}$, of all ground state contours is defined as

$$\mathscr{G} = \bigcap_{\Lambda \text{ finite}} G_\Lambda \tag{4.3}$$

We stress again that to each $\Gamma \in \mathscr{G}$ there correspond two ground state configurations of *spins*, $(\Gamma, +)$ and $(\Gamma, -)$, with the same Peierls contours, but differing by a global spin-flip.

A simple observation shows that two ground states cannot differ in energy by more than a surface term, in any finite box Λ. For, suppose Γ^1 and Γ^2 are in G and that

$$E(\Gamma^2_\Lambda) - E(\Gamma^1_\Lambda) > E(\partial\Lambda) \tag{4.4}$$

Then, consider a contour Γ, coinciding with Γ^1 inside Λ, with Γ^2 outside Λ, and which has some additional contours in $\partial\Lambda$ pasting together the two pieces. Then, obviously,

$$E(\Gamma_\Lambda) \leqslant E(\Gamma^1_\Lambda) + E(\partial\Lambda) < E(\Gamma^2_\Lambda) \tag{4.5}$$

which contradicts the assumption that $\Gamma^2 \in G_\Lambda$.

This result shows, in particular, that a unique ground state energy *density* exists, which is independent of the particular ground state considered.

We need some precise concepts to characterize ground state configurations, and in particular to quantify by "how much" two ground states differ. The first such concept we want to introduce is that of "*strong equivalence classes*" of ground states.

Roughly speaking, a strong equivalence class shall encompass ground states that are not distinguished by different boundary conditions in the limit when the size of the box Λ goes to infinity, and that are thus contained in the support of the same extremal Gibbs state, even as the temperature tends to zero.

To make this concept precise, consider the symmetric difference, $\Gamma^1 \Delta \Gamma^2$ of two ground states, Γ^1 and Γ^2. Clearly, since $\partial\Gamma^1 = \partial\Gamma^2 = \Phi$, the boundary $\partial(\Gamma^1 \Delta \Gamma^2)$ of $\Gamma^1 \Delta \Gamma^2$ in the interior of any box Λ is empty, i.e.,

$$\partial[(\Gamma^1 \Delta \Gamma^2)_\Lambda] \in \partial\Lambda \tag{4.6}$$

However, two entirely different situations may arise:

(i) $\Gamma^1 \Delta \Gamma^2$ is the disjoint[6] union of finite, closed surfaces,[7] S^α, $\alpha = 1, 2, 3,\ldots$. We then say that Γ^1 is *strongly equivalent* to Γ^2, $\Gamma^1 \overset{s}{\sim} \Gamma^2$.

[6] We allow closed surfaces to touch, however.

[7] We use here and elsewhere the generic name "surface" for $(d-1)$-complexes.

(ii) $\Gamma^1 \Delta \Gamma^2$ contains an infinite, connected, surface, which we denote by $[\Gamma^1 \Delta \Gamma^2]_\infty$. In this case, the restriction of $\Gamma^1 \Delta \Gamma^2$ to any (large enough!) box Λ has boundary in $\partial\Lambda$, and we say that $\Gamma^1 \Delta \Gamma^2$ has non-empty boundary at infinity. Γ^1 and Γ^2 are then called *strongly inequivalent.*

REMARKS

(i) As defined above, the notion of strong equivalence may in general have the deficiency not to be "*transitive.*" That is to say, there are situations where a state Γ^1 may be called equivalent to Γ^2, and likewise Γ^3 equivalent to Γ^2, whereas Γ^1 would not be equivalent to Γ^3 according to the above definition[8] It is then, however, desirable to put also $\Gamma^1 \sim^s \Gamma^3$. This way we can define the minimal transitive extension of our relation $\sim^s$, which is then appropriate to define strong equivalence classes. Formally this can be achieved by defining a family of relations $\sim^0$, $\sim^1$, $\sim^2$,..., where $\sim^0$ is the relation as defined above, and

$$\Gamma^1 \overset{i}{\sim} \Gamma^2$$

iff there exists a ground state Γ^3 such that

$$\Gamma^1 \overset{i-1}{\sim} \Gamma^3 \qquad \text{and} \qquad \Gamma^2 \overset{i-1}{\sim} \Gamma^3$$

Finally, Γ^1 is said to be equivalent to Γ^2, iff $\Gamma^1 \sim^n \Gamma^2$, for some n.

(ii) It is important to notice that for two ground states to be strongly inequivalent $\Gamma^1 \Delta \Gamma^2$ must have a boundary already in *finite* boxes. Thus if we construct two ground states by taking the limit of two finite volume ground states with two different sequences of boundary conditions, the resulting limiting states will be strongly inequivalent only if their symmetric difference is sufficiently localized, i.e., passes with finite probability through specified finite boxes Λ.

Strong equivalence has been defined so far for ground state *contours*. It can be extended to configurations of spins by attributing a contour at *infinity* to $(\Gamma, +)\Delta(\Gamma, -)$ and hence calling $(\Gamma, +)$ and $(\Gamma, -)$ inequivalent. It is easy to see that $\Gamma^1 \not\sim^s \Gamma^2$ implies that all the four configurations $(\Gamma^1, +)$, $(\Gamma^1, -)$, $(\Gamma^2, +)$, and $(\Gamma^2, -)$ are inequivalent, while $\Gamma^1 \sim^s \Gamma^s$ implies that $(\Gamma^1, +)$ is equivalent to exactly one of the two realization (Γ^2, σ), $\sigma = +$ or $-$.

Note that for arbitrary states $\Gamma \sim^s \Gamma^1$, $\Gamma' \sim^s \Gamma^2$, $(\Gamma \Delta \Gamma')_\infty = (\Gamma^1 \Delta \Gamma^2)_\infty$. In fact, this statement should be regarded as the precise definition of $(\Gamma^1 \Delta \Gamma^2)_\infty$.

[8] We are grateful to M. Aizenman for having pointed out this problem to us.

We see, that since $\partial[(\Gamma^1 \Delta \Gamma^2)_\infty]_\Lambda \neq \varnothing$ for all (large enough) boxes Λ, ground states in different strong equivalence classes are always distinguished by boundary conditions on any family of boxes which increase to $\mathscr{Z}^d$.

It is useful to introduce, furthermore, an alternative, somewhat more topological way of describing ground states, which will provide another appealing characterization of strong equivalence classes.

With a collection of contours, Γ, we may associate a function (or *co-chain*), Θ_Γ, on the bonds by setting

$$\Theta_\Gamma(i, j) = \begin{cases} -1, & \text{if } \langle ij \rangle^* \in \Gamma \\ +1, & \text{if } \langle ij \rangle^* \notin \Gamma \end{cases} \tag{4.7}$$

Taking the symmetric difference of two contours corresponds then to taking the product of the corresponding co-chains, i.e.,

$$\Theta_{\Gamma^1 \Delta \Gamma^2}(i, j) = \Theta_{\Gamma^1}(i, j) \cdot \Theta_{\Gamma^2}(i, j) \tag{4.8}$$

The boundary operation, too, has a simple interpretation in terms of exterior derivatives, i.e.,

$$\Theta_{\partial\Gamma} = d\Theta_\Gamma \tag{4.9}$$

Thus Eq. (4.6) may now be written as

$$d\Theta_{\Gamma^1 \Delta \Gamma^2} = 0 \tag{4.10}$$

which is to say that $\Theta_{\Gamma^1 \Delta \Gamma^2}$ is *closed.* But this, clearly, does not imply that $\Theta_{\Gamma^1 \Delta \Gamma^2}$ is *exact.* In fact, when $\Theta_{\Gamma^1 \Delta \Gamma^2}$ is exact, i.e.,

$$\Theta_{\Gamma^1 \Delta \Gamma^2} = d\Theta_D \tag{4.11}$$

for some subset D of $\mathscr{Z}^d$, we call Γ^1 and Γ^2 strongly equivalent.

While strong equivalence classes are useful to investigate the structure of Gibbs states at zero temperature, the situation at finite temperature is more subtle. There are two principal questions we must consider. One concerns the *spectrum of excitations* above a given ground state, the other the existence of *energy barriers* between two states.

Let us first turn to the question of the excitation spectrum. In general, guided by the *Pigorov–Sinai* theory,[18,19] one expects a ground state to be associated with an extremal Gibbs state only if it is *dominant*,[19] i.e., if the spectrum of excitations above it is maximal, in the following sense. Let $\rho_\Lambda(\Gamma, \delta)$, be the number of excitations above a ground state Γ that differ in energy from $E_\Lambda(\Gamma)$ by less than δ.[9] Let further the "*excitation density*," $\rho(\Gamma, \delta)$, be the coefficient of the term proportional to $|\Lambda|$ contributing to $\rho_\Lambda(\Gamma, \delta)$, i.e., the number of "connected" excitations of energy $\leqslant \delta$ in Λ. Then, Γ is called *dominant*, if

$$\int e^{-\alpha\delta}\, d\rho(\Gamma, \delta) \geqslant \int e^{-\alpha\delta}\, d\rho(\Gamma', \delta)$$

for all other ground states Γ', for some $\alpha < \infty$.

Note that the definition of "dominance" involves only the bulk quantity $\rho(\Gamma, \delta)$. Thus, in short-range models, e.g., the Edwards–Anderson model, two ground states, Γ^1 and Γ^2, may have a different excitation density only if the corresponding contours differ "everywhere," i.e., $\Gamma^1 \Delta \Gamma^2$ has Hausdorff dimension equal to the dimension of the lattice, d.

Let us suppose now that Γ^1 is a dominant ground state. Then, in the equivalence class of Γ^2 there is a state $\tilde{\Gamma}^2$ with contour given by $\tilde{\Gamma}^2 = \Gamma^1 \Delta [\Gamma^1 \Delta \Gamma^2]_\infty$. Thus $\tilde{\Gamma}^2$ is also dominant, if $[\Gamma^1 \Delta \Gamma^2]_\infty$ is a surface of codimension greater than zero, by the above arguments. We expect that this is the typical situation in the spin glass model (at least in dimensions greater than three), and that therefore to each equivalence class there belongs at least one dominant ground state.

Next we address the more difficult question of energy barriers between ground states. It is essential for the understanding of the properties of spin glasses. The exact definition of energy barrier that is appropriate is not entirely obvious or unique. We give a definition inspired by the droplet picture[(20)] and the Peierls argument[(21)] that yields reliable predictions in cases where the Pirogov–Sinai theory applies and in other cases where the correct answer is known.

Let (Γ^1, σ^1) and (Γ^2, σ^2) be two ground state configurations. We want to know the excess energy it costs to switch (Γ^1, σ^1) into (Γ^2, σ^2) within a region Λ. Thus, let, for $\Lambda' \supseteq \Lambda$, $\mathscr{C}_{12}^{\Lambda\Lambda'}$ be the set of all configurations (Γ, σ) such that $(\Gamma, \sigma)_\Lambda = (\Gamma^1, \sigma^1)_\Lambda$ and $(\Gamma, \sigma)_{\Lambda'^c} = (\Gamma^2, \sigma^2)_{\Lambda'^c}$. We define[10]

$$\Delta E_\Lambda[(\Gamma^1, \sigma^1), (\Gamma^2, \sigma^2)] \equiv \min\{\inf_{\Lambda'} \inf_{(\Gamma,\sigma)\in \mathscr{C}_{12}^{\Lambda\Lambda'}} (E(\Gamma_{\Lambda'}) - E(\Gamma^2_{\Lambda'})), 1 \leftrightarrow 2\} \tag{4.12}$$

Of course, we are interested in making $|\Lambda|$ large. The right finite quantity to extract is clearly the rate of growth of ΔE_Λ as $|\Lambda| \to \infty$. Thus we are prompted to define

$$\Delta((\Gamma^1, \sigma^1), (\Gamma^2, \sigma^2)) \equiv \liminf_{|\Lambda| \to \infty} \frac{1}{\ln|\Lambda|} \ln_+ [\Delta E_\Lambda[(\Gamma^1, \sigma^1), (\Gamma^2, \sigma^2)]] \tag{4.13}$$

[9] In Ref. 19 one deals with a quantity $n_\Lambda(\Gamma, \varepsilon)$ which is related to our ρ by $\rho_\Lambda(\Gamma, \delta) = \int_E^{E+\delta} d\varepsilon\, n_\Lambda(\Gamma, \varepsilon)$. The reason why we prefer to introduce ρ is that in our case we do not in general expect discrete spectrum.

[10] As D. Fisher pointed out to us, this quantity should perhaps be called a *twist modulus*.

Here

$$\ln_+ x = \begin{cases} \ln x, & \text{if } x \geqslant 1 \\ 0, & \text{if } x \leqslant 1 \end{cases} \tag{4.14}$$

The definition of Δ suggests the definition of "*weak equivalence classes*" of ground states, putting

$$(\Gamma^1, \sigma^1) \overset{w}{\sim} (\Gamma^2, \sigma^2) \qquad \text{iff} \quad \Delta((\Gamma^1, \sigma^1), (\Gamma^2, \sigma^2)) = 0 \tag{4.15}$$

From the analysis of known examples, we may reasonably expect a close relationship between weak equivalence classes and low-temperature extremal Gibbs states, *even though Δ does not take into account entropy contributions.* To get aquainted with this concept, let us mention the following examples.

(i) *One-dimensional Ising model, long-range interaction*

There are two ground states with empty contour $\Gamma_0 = \varnothing$, $(\Gamma_0, +)$, and $(\Gamma_0, -)$. If the interaction behaves as $J_{ij} \sim |i-j|^{-\alpha}$, we find

$$\begin{aligned} \Delta E_\Lambda[(\Gamma_0, +), (\Gamma_0, -)] &= \sum_{\substack{|i| < |\Lambda|/2 \\ |j| > |\Lambda|/2}} |i-j|^{-\alpha} \sim \sum_{0 < i < |\Lambda|/2} |i|^{-\alpha+1} \\ &\sim \begin{cases} \text{const}, & \text{if } \alpha > 2 \\ \ln|\Lambda|, & \text{if } \alpha = 2 \\ |\Lambda|^{2-\alpha}, & \text{if } \alpha < 2 \end{cases} \end{aligned} \tag{4.16}$$

and thus

$$\Delta[(\Gamma_0, +), (\Gamma_0, -)] = \begin{cases} 0, & \text{if } \alpha \geqslant 2 \\ (2-\alpha), & \text{if } \alpha < 2 \end{cases} \tag{4.17}$$

Now, it is known that, for $\alpha \leqslant 2$, there exist at least two disjoint extremal Gibbs states at low enough temperature,[22,23] while for $\alpha > 2$ the Gibbs state is unique at all positive temperatures. Hence, in these models the condition $\Delta[(\Gamma_0, +), (\Gamma_0, -)] > 0$ is sufficient to conclude the existence of two disjoint Gibbs states. The model with $\alpha = 2$ is a borderline case with logarithmically growing energy barrier. Our criterion may therefore not give the correct prediction in this case. It is now known[23] that it has a transition, but this situation is very delicate and changes once Ising spins are replaced by N-vector ($N \geqslant 2$) spins. Our criterion also makes correct predictions for the N-vector model, as follows from the results in Ref. 24.

(ii) *Interfaces in d-dimensional nearest-neighbor Ising models*

It is known that in the nearest neighbor Ising model there exist disjoint Gibbs states that differ by "*interfaces*" of dimension[11] down to two.[25,26] Now note that, if there are two ground states (Γ^1, σ^1) and (Γ^2, σ^2) such that $\Gamma^1 \Delta \Gamma^2$ is made of two "surfaces" of dimension D some finite distance apart, then

$$\Delta[(\Gamma^1, \sigma^1), (\Gamma^2, \sigma^2)] = \frac{D-1}{d} \tag{4.18}$$

implying that such surfaces do not fluctuate once their dimension is bigger than one, thus yielding the correct prediction.

Observe that in this case we could refine our criterion by taking the entropy contributions into account: For two states (Γ^1, σ^1) and (Γ^2, σ^2) whose symmetric difference is a surface of dimension D, $\Delta[(\Gamma^1, \sigma^1), (\Gamma^2, \sigma^2)]$ must be bigger or equal to $(D-1)/d$, since this is the size of the entropy contribution. Evidently, this stronger criterion is automatically satisfied and the result does not change.

We expect that the last observation will also apply in the spin glass: either Δ will be zero, or, if it is positive, it will always be at least as big as the entropy term (at least if spin-flip symmetry is broken).

The energy barriers defined above are in general quite difficult to calculate. It is thus useful to introduce a simpler and more geometric alternative, which is more easily verified, and which we expect to be equivalent to the one above, *if the global spin-flip symmetry is broken.*

Let Γ^1 and Γ^2 be two ground state configurations,[12] and $A_\Lambda(\Gamma^1, \Gamma^2)$ the minimal area of the holes of $([\Gamma^1 \Delta \Gamma^2]_\infty)_\Lambda$ in $\partial\Lambda$. Then let

$$\bar{\Delta}(\Gamma^1, \Gamma^2) = \lim_{|\Lambda| \to \infty} \frac{1}{\ln|\Lambda|} \ln_+ (A_\Lambda(\Gamma^1, \Gamma^2)) \tag{4.19}$$

We clearly have the inequality

$$\bar{\Delta}(\Gamma^1, \Gamma^2) \geqslant \Delta(\Gamma^1, \Gamma^2) \tag{4.20}$$

If the spin-flip symmetry is broken (we expect this to be the case in dimensions $\geqslant 3$), we even expect that equality holds for almost all ground states. Namely, consider a ground state $(\Gamma, +)$ and its spin-flipped counterpart $(\Gamma, -)$. We expect that

$$\bar{\Delta}((\Gamma, +), (\Gamma, -)) = \frac{d-1}{d} \tag{4.21}$$

[11] Dimension in this context is understood to mean the number of dimensions in which the object under consideration has infinite extent.

[12] We suppress henceforth the global spin indicator σ if no confusion can arise.

If $\Delta((\Gamma, +), (\Gamma, -))$ is smaller than this value, one might expect a restoration of the spin-flip symmetry (the entropy, i.e., the number of possible droplets, becomes dominant over the energy barrier). If on the other hand $\Delta((\Gamma, +), (\Gamma, -)) = (d-1)/d$, it is natural to expect that the energy barrier is in general proportional to the area of the holes in $[\Gamma^1 \Delta \Gamma^2]_\infty \cap \partial \Lambda$, and that thus equality holds in (4.20).

An interesting feature of $\bar{\Delta}$ that is worth mentioning is that since

$$A_\Lambda(\Gamma^1, \Gamma^3) \leqslant A_\Lambda(\Gamma^1, \Gamma^2) + A_\Lambda(\Gamma^2, \Gamma^3) \tag{4.22}$$

$\bar{\Delta}$ defines an ultrametric, i.e., $\bar{\Delta}$ is a metric and further

$$\bar{\Delta}(\Gamma^1, \Gamma^3) = \max\{\bar{\Delta}(\Gamma^1, \Gamma^2), \bar{\Delta}(\Gamma^2, \Gamma^3)\}, \qquad \text{if } \bar{\Delta}(\Gamma^1, \Gamma^2) \neq \bar{\Delta}(\Gamma^2, \Gamma^3) \tag{4.23}$$

This fact may be of particular interest in view of the discussion of ultrametric topologies on the space of extremal Gibbs states in the *Sherrington–Kirkpatrick* model in the context of "*replica symmetry breaking*" by Mézard *et al.*[8] The ultrametric discussed there, the so-called "*overlap parameter*" $q_{\alpha\beta}$, is rather different from our Δ but may, however, also be given a natural geometric interpretation in the Edwards–Anderson model. It is defined as

$$q_{\alpha\beta} \equiv \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta \tag{4.24}$$

where α and β label two extremal Gibbs states. Supposing, for the time being, that α and β have support concentrated on two ground states, Γ^α and Γ^β, we find

$$q_{\alpha\beta} = \lim_{|\Lambda| \to \infty} \left\{1 - \frac{\mathrm{vol}(\Gamma^\alpha \Delta \Gamma^\beta)}{|\Lambda|}\right\} \tag{4.25}$$

i.e., $q_{\alpha\beta}$ measures the number of spins that must be flipped to deform Γ^α into Γ^β. It is somewhat disturbing that this quantity does not seem to discern the global topological features of the symmetric differences, and it appears thus on the level of ground states not to be the appropriate metric. (We note, however, that if $|J_{ij}|$ can fluctuate, $q_{\alpha\beta}$ only involves global differences!) The quantity $q_{\alpha\beta}$ appears to be, in any case, relevant for dynamical aspects, more than for equilibrium properties.

Finally, it may be noteworthy that the above spaces of equivalence classes are naturally endowed with further mathematical structure. Namely, for a fixed frustration network Φ, we may consider the set $G(\Phi)$, defined as

$$G(\Phi) = \{\Gamma^1 \Delta \Gamma^2 \mid \partial \Gamma^1 = \Phi, \partial \Gamma^2 = \Phi\} \tag{4.26}$$

Clearly, $G(\Phi)$ forms a group, which is most easily checked by considering the associated co-chains $\Theta_{\Gamma^1 \Delta \Gamma^2}$. The group structure on $G(\Phi)$ induces a group structure also on the associated sets of (weak or strong) equivalence classes. Denote by $[\Gamma]$ the strong equivalence class of Γ, and let

$$[\Gamma^1 \Delta \Gamma^2] = \{\tilde{\Gamma}^1 \Delta \tilde{\Gamma}^2 \mid \tilde{\Gamma}^1 \in [\Gamma^1], \tilde{\Gamma}^2 \in [\Gamma^2]\} \tag{4.27}$$

Then

$$G^s(\Phi) = \{[\Gamma^1 \Delta \Gamma^2] \mid \partial\Gamma^1 = \Phi, \partial\Gamma^2 = \Phi\} \tag{4.28}$$

The elements of the set $G^s(\Phi)$ are homology classes represented by the forms $\Theta_{\Gamma^1 \Delta \Gamma^2}$.

The same construction can obviously be done with weak equivalence classes, thereby generating a group $G^w(\Phi)$.

5. LOW-ENERGY EXCITATIONS AND LONG-RANGE CORRELATIONS

In this section we discuss the spectrum of excitations near a given ground state, i.e., local fluctuations with small energy at fixed boundary conditions. There are two facts we want to establish. First, the spectrum of such fluctuations extends down to zero energy. Second, and most importantly, in three and more dimensions there are, with *large* probability, low-energy excitations that cause long-range correlations between the values of spins at two sites x and y with $|x-y|$ arbitrarily large.

The fluctuations of lowest possible energy are of course due to possible ground state degeneracies. Whether such degeneracies are present or not depends on the form of the J-distribution. If it is concentrated on a set of discrete values (e.g. ± 1), then there are many degenerate ground states, and the number of them, as we shall see, grows exponentially with the volume of the box we consider. But if the J-distribution is continuous, exact degeneracies are arbitrarily unlikely, simply since two real numbers are unequal, with probability one.

This difference will, however, not be very important, except at $T=0$, where true degeneracies can give rise to residual entropy and the like. As soon as T is nonzero, the important quantities are the number $\rho_\Lambda(\delta)$ of states in a small, but finite interval, δ, of energies above the ground state energy. For this quantity we again find

$$\rho_\Lambda(\delta) \sim \rho(\delta)|\Lambda| + O(|\Lambda|^2)$$

with $\rho(\delta) \to 0$ as $\delta \to 0$. Here $\rho(\delta)$ is the number of *connected* configurations of energy $\leqslant \delta$ above the ground state energy, per unit volume.

The degeneracy of the ground states in the discrete case was first proven by Avron *et al.*[14] for the two-dimensional model. They showed that with finite probability there exists, in a small region Λ_0 of the lattice, an arrangement of frustrated plaquettes that supports at least two minimal contours, no matter what the external configuration is like. If the volume of this region is $|\Lambda_0|$ and the probability for this configuration of frustrations is p, then, in a box of volume $|\Lambda|$, one expects $p|\Lambda|/|\Lambda_0|$ such configurations, and there are thus $(2^{p/|\Lambda_0|})^{|\Lambda|}$ degenerate ground states.

It is very easy to show that the same holds in three dimensions. A possible arrangement consists of three 2×2 frustration loops placed at a distance one behind each other, and enclosed in a $5\times 5\times 5$ box which is otherwise free of frustration (see Fig. 4).

Obviously this occurs with a finite probability (which we do not care to calculate explicitly) if $0<x<1$. For all boundary conditions that one may impose on this box, the ground state contour within it is twofold degenerate.

The same argument shows that in the case of a continuous distribution, $\rho_\Lambda(\delta)\geqslant \mathrm{const}|\Lambda|$, for any $\delta>0$. One only has to observe that with finite probability the values of the $|J_{ij}|$ in the above arrangement are such that the two states differ in energy only by $\delta>0$ and one of them is the ground state.

Unfortunately such simple considerations only give rather crude results. In particular, a more precise prediction of the behavior of the den-

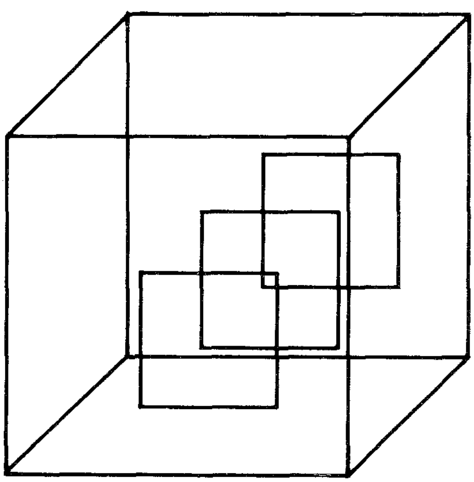

Fig. 4. Three frustration loops enclosed in an otherwise unfrustrated box that supports at least two degenerate ground state configurations.

sity of "connected" states with energy near the ground state energy appears difficult to obtain.

A question we are most interested in is whether there are excitations of low energy that can correlate spins which are very far apart. To see how such correlations may come about, expand an extremal Gibbs state $d\mu$ formally into a series of measures concentrated on particular excitations, i.e.,

$$d\mu = \frac{1}{Z} \sum_i e^{-\beta E^i} \, d\mu^i \tag{5.1}$$

where $d\mu^i$ is concentrated on a class of configurations i, with energy E^i above the ground state energy, and Z is the appropriate normalization factor. Suppose there is a configuration i such that the spins σ_x and σ_y are flipped in i relative to the ground state, and which cannot be decomposed into two disjoint excitations i_x and i_y that have one spin flipped, but not the other, and for which $E^i = E^{i_x} + E^{i_y}$. Then, this configuration contributes a term to the connected two-point function $\langle \sigma_x ; \sigma_y \rangle$ which is of order $\exp(-\beta E^i)/Z$. In a ferromagnetic Ising model, one easily sees that the energy of such a configuration is at least $O(|x-y|)$, and hence we get exponential decay at small temperatures. In the three-dimensional spin glass, due to the presence of a dense frustration network, the energy of such excitations can be much lower, and thus there may not be exponential decay of correlations.

Of course, to arrive at this conclusion, we must also assume that there are no other terms of the same magnitude in which the relative orientation of the two spins σ_x and σ_y is reversed, and which could therefore cancel the original term and destroy the long-range correlation. This would arise if the global spin-flip symmetry were unbroken. Pictorially speaking, if the spin-flip symmetry is unbroken, correlations over long distances that could arise from coherent spin-flips within a domain Ω are destroyed by incoherent spin-flips inside large droplets $D \subset \Omega$. However, *if* the spin-flip symmetry is *broken*, such *additional* flips cost too much energy, i.e., have small probability, and hence correlations persist. More precisely, if the spin-flip symmetry is broken, the probability of finding a droplet $D \subset \Omega$ surrounding x or y, but not both, with the property that the spins within D may be flipped without significantly changing the energy of the configuration, is strictly less than 1, uniformly in $|x-y|$.

As we will argue in the next section, global spin-flip symmetry appears to be unbroken in *two dimensions*, and hence we do not expect long-range correlations there. In *three and more dimensions*, however, the two-dimensional argument breaks down, and we will argue, though not prove, that the global spin-flip symmetry *is* broken in this case. This conjecture is supported by numerical evidence.[4]

It is easy to prove rigorously that excitations of finite energy correlating σ_x and σ_y exist with probability $O(\exp(-k|x-y|))$, using arguments very similar to those used to prove the ground state degeneracy, *for all* $x \in (0, 1)$. However, this is clearly not what we need; it obviously does not imply that the averaged two-point functions do not have exponential decay. (In fact, this conclusion would be false for low density of frustration, in which case we know the correlation length to be finite for small temperatures[14]!)

For x near 1/2, however, we expect that such excitations exist with much larger probability (i.e., with a probability that does *not decay exponentially* with the distance between x and y!), and in the remainder of this section we give a heuristic argument supporting their existence. More precisely, we argue that with "large" probability there exists an environment of frustration loops for which there is a ground state and excited states with energies only a little above the ground state energy and that differ by having all spins flipped within a tube of arbitrary length R. To be precise, we must show that neither the probability for this decays exponentially with R, nor does the energy of the excitation grow linearly with R.

In order to do this we return to the "cylinder" events that we have constructed at the end of Sec. 3. Consider a tube of length R and cross-section $L \times L$. We may cover it with cylinders of the form $L \times L \times \lambda$ made of double layers. Now we have shown that the probability that there is no frustration loop winding around such a cylinder within the double layers is less than $4L\exp(-m\lambda)/m$. The individual cylinder events are independent if $x = 1/2$, and almost so if x is near 1/2. Thus the probability that n consecutive cylinder events fail to occur somewhere on the tube is bounded by

$$\frac{R}{\lambda}\left(\frac{4L}{m}e^{-m\lambda}\right)^n \tag{5.2}$$

Therefore, the probability that the minimal area between two consecutive loops on the tube is larger than the minimal area across one loop is bounded above by

$$\frac{R}{\lambda}\left(\frac{4L}{m}e^{-m\lambda}\right)^{L/4\lambda} = \frac{R}{\lambda}\exp\left\{\frac{L}{4}\left(\frac{1}{\lambda}\ln\frac{4L}{m} - m\right)\right\} \tag{5.3}$$

If λ is chosen such that

$$\lambda > \frac{1}{m}\ln\frac{4L}{m} \tag{5.4}$$

this probability can be made arbitrarily small by choosing L somewhat large, i.e.,

$$L > \frac{\ln(R/\lambda)}{m - (1/\lambda)\ln(4L/m)} \tag{5.5}$$

That is, λ must be of the order of $\ln L = \ln \ln R$. In the absence of further frustration loops, the ground state configuration of such a tube then simply consists of minimal surfaces spanned between consecutive pairs of cylinders. (See Fig. 5.)

Suppose now we flip all the spins within the tube. The resulting Peierls contours are as drawn in Fig. 6.

What is the typical energy difference between these two configurations? First of all, we have to add the two "lids" at the ends of the tube, which costs an energy of order $2L^2$. Second, we interchange the covered with the uncovered areas on the tube. Since the areas between two cylinders are essentially independent random variables, for large R we

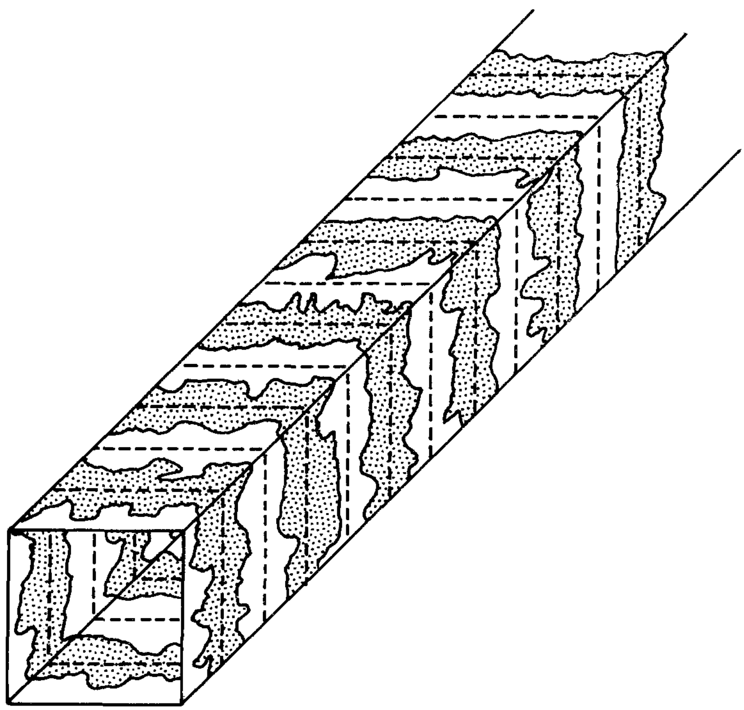

Fig. 5. Ground state configuration of Peierls contours (shaded area) on a tube.

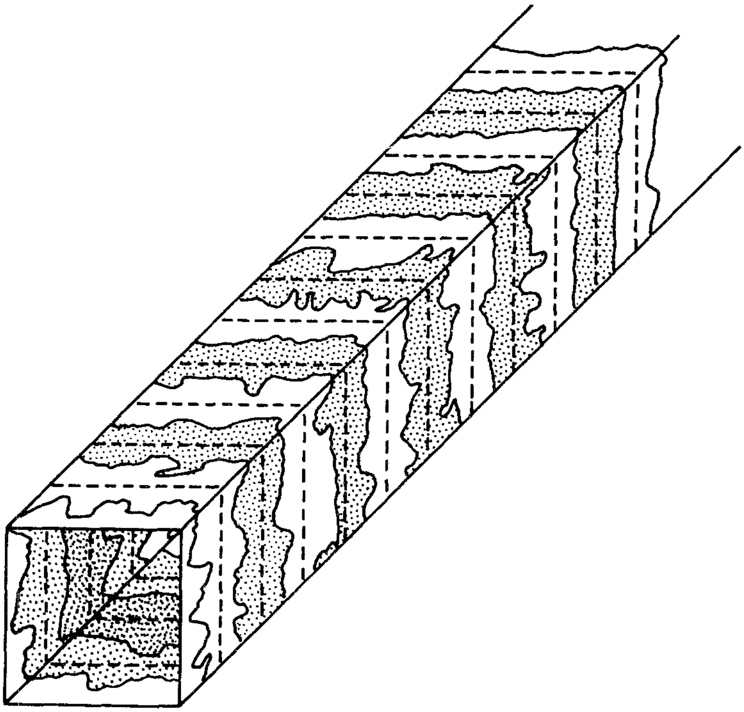

Fig. 6. Resulting contour after all spins within the tube have been flipped.

invoke the central limit theorem to estimate this difference. The covariance of the single pieces of surface is roughly $4L\lambda/c$, where $c = m\lambda - \ln(4L/m)$.

The difference between the covered surfaces, and thus the energy difference, is thus a Gaussian random variable with mean $2L^2$ and variance $\sqrt{2R/\lambda}\, 4L\lambda/c$. Thus the probability, P, that flipping all the spins within the tube costs an energy between E and $E + \Delta E$ is

$$P = \frac{\Delta E}{\sqrt{\pi}} \frac{1}{\sqrt{2R\lambda}\,(4L/c)} \exp\left\{-\frac{(E-2L^2)^2}{2R\lambda(4L/c)^2}\right\} \tag{5.6}$$

$$\approx \Delta E \frac{1}{\sqrt{2R}} \exp\left\{-\frac{(E-(\ln R)^2)^2}{2R}\right\} \tag{5.7}$$

[In (5.6) L and λ are understood to be chosen so as to just satisfy conditions (5.4) and (5.5). Equation (5.7) is the result of this procedure, up to subleading logarithms of R.]

This is indeed the result we were looking for.

The above considerations did not take into account that there are in general many further frustration loops present. These clearly modify the ground state configurations and we have to ask whether the above picture survives in its essence.

Clearly further frustration loops on the surface of the tube cannot have a negative effect. The most that can happen is that the minimal surface takes a checkerboard-like structure, splitting into even more and smaller pieces, still covering roughly half of the total surface of the tube. The same heuristic arguments then hold, and the expected energy difference will be even smaller (see Fig. 7).

The existence of other large loops close to the ones on the tube inside or outside of the tube would possibly favor fluctuations concentrated on a somewhat deformed tube, but hardly spoil the overall picture; see also Sec. 7.

Thus it is very plausible that, in dimensions $\geqslant 3$, there exist, with probability decaying more slowly than exponentially, excitations of small energy that cause long-range correlations.

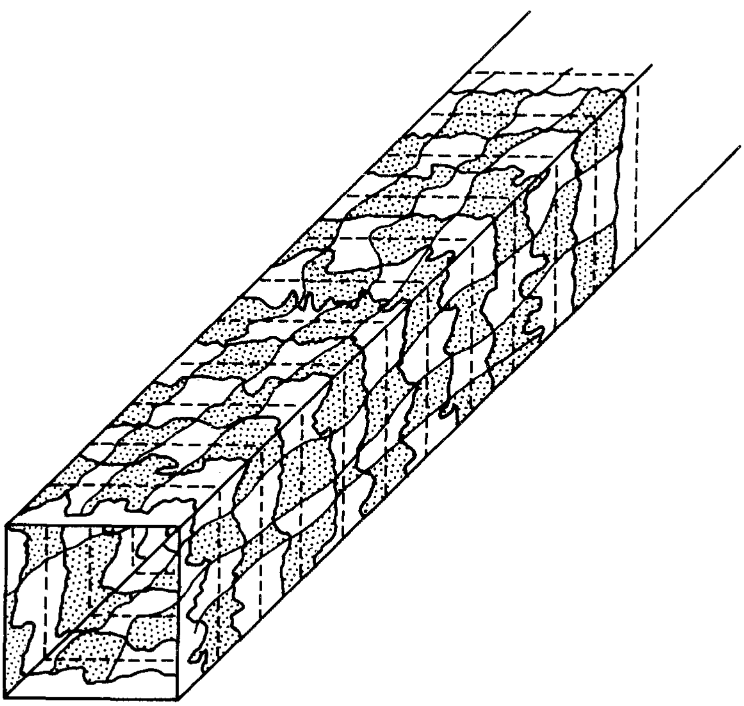

Fig. 7. Ground state configuration on the tube in the presence of further frustration loops.

Let us now summarize this discussion. Let Γ be a dominant ground state, and let $[\Gamma]$ denote the class of all those families of contours, Γ', for which $\bar{\Delta}(\Gamma, \Gamma') = 0$. Let Ω_{0y} be some finite region in $\mathscr{Z}^d$, $d \geqslant 3$, containing two sites, 0 and y. Let Γ' be the configuration of contours obtained from Γ by flipping all the spins inside Ω_{0y} (and suppose $\Gamma' \in [\Gamma]$). We define

$$\Delta E(\Omega_{0y}, \Gamma) \equiv E(\Gamma') - E(\Gamma) \geqslant 0$$

The heuristic calculations leading to (5.6) and (5.7) suggest that

$$dP_{0y}(E) \equiv \mathrm{Prob}\{\Omega_{0y} \mid \Delta E(\Omega_{0,y}, \Gamma) \in [E, E + dE]\}$$
$$> \frac{1}{\sqrt{|y|}} \exp\left\{-\frac{(E - (\ln|y|)^2)^2}{2|y|}\right\} dE$$

Let P_{flip} be the probability that there is a spin-flip inside a region containing 0 or y, but not both. Let β be large, and let $\langle(\cdot)\rangle_\beta$ be a Gibbs state in the "vicinity" of Γ, (e.g., the support of $\langle(\cdot)\rangle_\beta$ is contained in $[\Gamma]$; see (5.1)). The mathematical structure of low-temperature expansions suggest that, for β very large,

$$|\langle\sigma_0; \sigma_y\rangle_\beta| \sim (1 - P_{\text{flip}}) \int_0^\infty e^{-\beta E}\, dP_{0y}(E)$$
$$> (1 - P_{\text{flip}}) \frac{\text{const}}{\sqrt{|y|}} \tag{5.8}$$

Hence $\xi(\beta)$ is *divergent* if the spin-flip symmetry is broken, i.e., if $P_{\text{flip}} < 1$.

6. EQUIVALENCE CLASSES AND GIBBS STATES

In the last section we have seen that a characteristic feature of the low-temperature phase of the spin glass (in $d \geqslant 3$) is the divergence of the correlation length, *provided the global spin-flip symmetry is broken.* This divergence was caused by low-lying excitations above a given groundstate. In the present section we address the question of the structure of the space of Gibbs states themselves. Here we are primarily interested in ground states distinguished by different boundary conditions, since they might give rise to different disjoint Gibbs states. The tools for this investigation have been prepared in Sec. 4.

We split our discussion in two parts. First, we consider the two-dimensional case and argue that here the spin-flip symmetry is unbroken, for $x \approx 1/2$, and that consequently there exists a unique Gibbs state, at all temperatures. We also discuss the possibility of spin-flip symmetry breaking in higher, and particularly three, dimensions. We indicate why the two-dimensional argument does not apply and that, thus, in particular, in view of recent numerical results,[4] it does seem plausible that the global spin-flip symmetry is broken at low temperatures in three and more dimensions.

In the second part, we assume broken spin-flip symmetry and discuss the ensuing consequences for the structure of low-temperature Gibbs states in three and more dimensions.

6.1. Two Dimensions and Spin-Flip Symmetry Breaking

In two dimensions Φ is a collection of sites in the dual lattice, and a ground state configuration is a set of shortest connections between pairs of them.

Of course "short" is to be understood here with regard to the distance defined through the energy function $E(\Gamma)$, which depends on the moduli of the J_{ij}. In the case where $|J_{ij}| \equiv 1$, this distance coincides with the usual "Manhattan" distance on the lattice. As long as typical individual lines in Γ are short—and from the properties of Φ exhibited in Sec. 3 we know this always to be the case—there will not be much difference between the two concepts, and we will mostly think in terms of the $|J_{ij}| \equiv 1$ case.

For small densities of frustrated plaquettes [i.e., x or $(1-x)$ small], a typical configuration of them consists of pairs of two frustrated plaquettes with comparatively large distances between different pairs. The minimal Peierls contours will therefore consist of short "strings" connecting the two partners in a pair. In this situation, the standard Peierls argument still proves the existence of a ferromagnetic (resp. antiferromagnetic) phase transition. See, e.g., Avron *et al.*[14] In fact, numerical simulations indicate that a ferromagnetic phase exists as long as[27]

$$x < 0.12 \pm 0.04 \tag{6.1}$$

(The rigorous proof of Ref. 14 works up to about half that value.)

As x increases, the density of contour islands grows. Finally, frustrated plaquettes start to star-percolate, and Peierls contours are abundant all over the lattice. Also, their assignment to pairs of frustrated plaquettes becomes highly ambiguous. However, due to the high density of frustrated plaquettes, essentially no contours longer than a few lattice units appear in a ground state. In fact, this statement can be quantified. The density of Peierls contours, α, in a ground state of the two-dimensional model is known from numerical simulations[27]; there exist also rigorous lower bounds.[28] For the $\{J_{ij} = \pm 1\}$-distribution, the numerical value is, for $x = 1/2$,

$$\alpha = 0.15 \pm 0.0025 \tag{6.2}$$

Given the density of sites dual to frustrated plaquettes of 1/2, there are $x/2\alpha \approx 5/3$ boundary points per bond in the Peierls contour. Such a high density implies an abundance of *isolated single bonds.* Even if we assume that only clusters of one and two bonds exist, it follows that the number of clusters with one bond exceeds that of the two-bond clusters by a factor of seven!

We now proceed to argue that this causes the global spin-flip symmetry to remain unbroken in two dimensions. Note that in fact this is the only thing we have to worry about in two dimensions. Though in general strong equivalence classes are characterized by lines in their symmetric differences, such lines always fluctuate at positive temperatures,[26] even in the frustrated system. In frustrated systems fluctuations around straight lines are enhanced and occur already at zero temperature.[29] Therefore at most two Gibbs states, differing by a global spin-flip, would be conceivable.

Let us consider two ground states, $(\Gamma, +)$ and $(\Gamma, -)$, related by a global spin-flip. To deform $(\Gamma, +)$ into $(\Gamma, -)$ within a box Λ, we have to draw a line, λ, enclosing Λ and flip all the spins enclosed by λ (see Fig. 8).

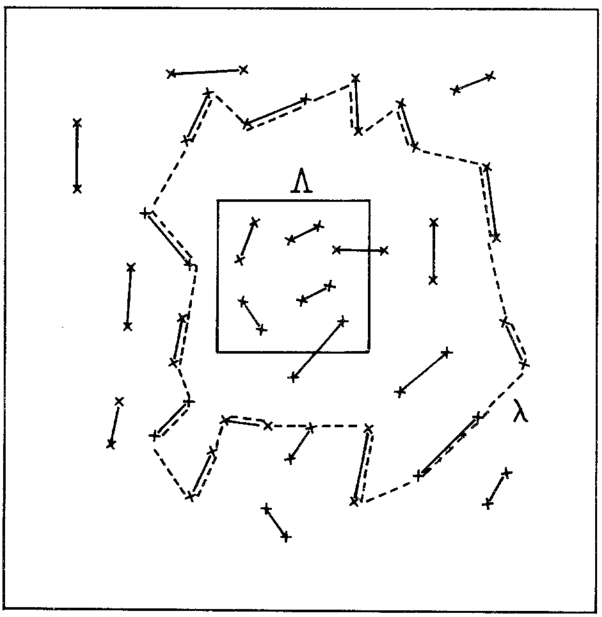

Fig. 8. A loop λ (dotted line) in a ground state configuration of Peierls contours (solid lines). Flipping all spins within λ, and hence in Λ, costs practically no energy at all.

The excess energy, $E(\lambda)$, associated with such a line is clearly

$$E(\lambda)=2 \sum_{\substack{\langle ij\rangle \in \lambda \\ \langle ij\rangle \notin \Gamma}} |J_{ij}|-2 \sum_{\substack{\langle ij\rangle \in \lambda \\ \langle ij\rangle \in \Gamma}} |J_{ij}| \tag{6.3}$$

In order to decide whether such a deformation occurs, one would want to know whether $\sum_\lambda e^{-\beta E(\lambda)}$ is large or small. If it tends to zero as the box Λ increases, the spin-flip symmetry will be broken (this is the Peierls argument); otherwise, we expect it to remain unbroken.

First of all, we will now argue that in the spin glass case, the only important contributions to this sum come from lines whose energy is practically zero. To understand this, we consider the average contribution of a single line λ. We may write

$$\overline{e^{-\beta E(\lambda)}}=\sum_{E\geqslant 0} \operatorname{Prob}(E(\lambda)=E)\, e^{-\beta E} \tag{6.4}$$

$\operatorname{Prob}(E(\lambda)=E)$ depends in general on the particular ground state considered and is not at all easy to calculate. It seems, however, reasonable to approximate this quantity by replacing Γ by a configuration of bonds occupied according to a Bernoulli bond percolation process with density α.[13] Then,

$$\operatorname{Prob}(E(\lambda)=E)=\binom{L}{(L-E/2)/2} \alpha^{(L-E/2)/2}(1-\alpha)^{(L+E/2)/2} \tag{6.5}$$

L being the length of λ. But then

$$\begin{aligned}
&\sum_{E/4=0}^{[L/2]} \binom{L}{(L-E/2)/2} \alpha^{(L-E/2)/2}(1-\alpha)^{(L+E/2)/2}\, e^{-\beta E} \\
&\quad=\sum_{k=0}^{[L/2]} \binom{L}{k} \alpha^k(1-\alpha)^{L-k} e^{-\beta(L-2k)} \\
&\quad=(1-\alpha)^L e^{-\beta L} \sum_{k=0}^{[L/2]} \binom{L}{k}\left[\frac{\alpha}{(1-\alpha)}\, e^{4\beta}\right]^k
\end{aligned} \tag{6.6}$$

The last sum is essentially equal to its last term, provided β is somewhat large, so that

$$\frac{\alpha}{(1-\alpha)}\, e^{4\beta}>1 \tag{6.7}$$

[13] We limit the discussion to the $|J_{ij}| \equiv 1$ case.

For such β,

$$\overline{e^{-\beta E(\lambda)}} \approx (\alpha(1-\alpha))^{[L/2]} \binom{L}{[L/2]} \approx \frac{1}{\sqrt{L}} (2\sqrt{\alpha(1-\alpha)})^L \tag{6.8}$$

independent of β. We see therefore that for a rather large range of temperatures the question of the breaking of the spin-flip symmetry is solely related to the probability of the existence of large lines associated with zero energy. If those fail to exist, a finite positive temperature is needed before the symmetry can be restored. This situation is quite typical for the spin glass, and it is the basic rationale for our definition of weak equivalence classes through purely energetic considerations.

We are now left to decide whether the result (6.8) implies the abundance of large zero-energy lines or not. This appears to be a rather interesting problem of percolation theory in itself, which has to our knowledge not received attention in the literature. Note that simply summing (6.8) over all lines λ will not give a good result, since the energies of different lines are not independent random variables. A rather reasonable criterion seems to be given by comparing this question with normal percolation. In a Bernoulli ensemble of bonds occupied with density p, the probability for a loop λ to be made entirely of occupied bonds is simply $p^{|\lambda|}$. In this case, such loops enclosing arbitrarily large boxes exist with probability one above the percolation threshold, i.e., if $p > 1/2$, while with probability one they do not exist for $p < 1/2$. This leads us to conjecture that large zero-energy loops will exist, provided

$$2\sqrt{\alpha(1-\alpha)} > \frac{1}{2} \tag{6.9}$$

that is, if

$$\alpha > \alpha_c = \frac{2-\sqrt{3}}{4} \approx 0.0675 \tag{6.10}$$

Since at, $x = 1/2$, $\alpha = 0.15$, this clearly implies that spin-flip symmetry is restored at all temperatures for x near $1/2$.[14] Furthermore, taking these numbers somawhat more seriously than we maybe should, we may estimate the critical density above which this should take place. From our previous discussion it appears not unreasonable to estimate the density of groundstate contours as proportional to the density of frustrated plaquettes, with the proportionality constant inferred from the values at $x = 1/2$.

[14] Note that even a rigorous lower bound for α is larger than α_c (Ref. 28).

The result of this procedure is

$$x_c \approx 0.0905 \tag{6.11}$$

a value which agrees remarkably well with numerical values, (6.1), for the breakdown of ferromagnetism!

It is interesting to see what becomes of this argument in three dimensions. There, the numerical estimate for α is[27]

$$\alpha \approx 0.21 \tag{6.12}$$

The corresponding percolation threshold for the existence of infinite closed surfaces of occupied plaquettes is known to be about $p_c = \frac{3}{4}$.[16] Thus the three-dimensional analog of (6.9) yields the estimate

$$2\sqrt{\alpha_c(1-\alpha_c)} = \frac{3}{4}$$

or

$$\alpha_c = \frac{1}{2} - \sqrt{\frac{7}{64}} \approx \frac{1}{6} \tag{6.13}$$

So, in this case, α_c and α are rather close together, and rather small errors can change the conclusion. In fact, we expect that the approximation of the Peierls contour by a Bernoulli ensemble is less reliable in three dimensions, and tends to overestimate the probability for surfaces of small energy. The reason for this is that in three dimensions, the *percolating* frustration network is the boundary of a Bernoulli plaquette ensemble at density $x \gg \alpha$. The ground state Peierls contour is thus forced to form *bigger and fewer* clusters than would exist in a Bernoulli ensemble at density α. Therefore, the distribution of the plaquettes belonging to the Peierls contour is less uniform, and exact cancelations, producing surfaces with low energy, are less likely. Thus, although this argument does not suffice to demonstrate breaking of the spin-flip symmetry in three dimensions, for $x = 1/2$, it shows that this is not unplausible. In particular, it shows why the behavior in two and three dimensions can be different. This would also be in agreement with results of recent numerical simulations.[4] See also Sec. 7.

6.4. Three and More Dimensions

We now assume that the global spin-flip symmetry is broken in three and more dimensions, and discuss some consequences for the structure of the low-temperature Gibbs states. The situation regarding the possible ground states is rather involved in higher dimensions. In contrast to, say, an unfrustrated Ising model, it is clearly not possible to find explicitly ground state contours for a given frustration network, Φ. As we have discussed above, this would correspond (for the $J_{ij} = \pm 1$ case, say) to finding the minimal surfaces with given random boundary Φ. Such problems, known as "first-passage problems" in mathematics, are known to be, even numerically for finite systems, extremely hard to solve.

Luckily, however, even without solving this problem exactly, we can give some general characterizations of the ground state contours, which allow us to draw some nontrivial conclusions, and to understand some features of the spin glass phase at least qualitatively. Our first observation is a corollary of our Lemma 3.2:

> If the density of negative J_{ij}, x, is close enough to 1/2, then, with probability one, any ground state contour, Γ, in dimensions $d \geqslant 3$, contains an infinite, star-connected cluster of $(d-1)$-cells.

Furthermore, we argue that there exist many ground states, corresponding to different strong equivalence classes, that are qualitatively indistinguishable, i.e., in particular, have essentially the same ground state energy and the same symmetry properties.

To see this, recall that two different strong equivalence classes correspond to different boundary conditions. Consider a box, Λ, and the intersection $\Phi_{\partial\Lambda}$ of Φ with the boundary of this box, $\partial\Lambda$. Now, *$\Phi_{\partial\Lambda}$ represents exactly a $(d-1)$-dimensional frustration network on $\partial\Lambda$*! Thus, boundary conditions are nothing but contours of a $(d-1)$-dimensional spin glass that lives on the boundary of Λ.

Clearly, boundary conditions that give rise to *ground states* (in the d-dimensional system) *with lowest energy* correspond, at least roughly, to a ground state contour of the $(d-1)$-dimensional system on $\partial\Lambda$. Thus, in three dimensions, the boundary conditions are contours of a two-dimensional spin glass on the surface of the box Λ, and the corresponding ground state will have minimal energy, if this contour just pairs sites and does not contain any extra loops.

It is not unreasonable to expect that, if the lengths of two contours $\Gamma^1_{\partial\Lambda}$ and $\Gamma^2_{\partial\Lambda}$ are equal, then the corresponding ground state energies $E_\Lambda(\Gamma^1)$ and $E_\Lambda(\Gamma^2)$ will also be close to each other.

We have just seen that in two dimensions states of essentially equal energy, but differing by a large contour λ, do exist. Therefore, in the three-dimensional model, we can have boundary conditions differing by a large loop—winding around the box Λ, say—that give rise to ground states Γ^1 and Γ^2 with very similar energy, whereas $[(\Gamma^1 \Delta \Gamma^2)_\infty]_\Lambda$ is a surface with boundary λ, spanning across Λ.

It is obvious that this argument propagates into higher dimensions.

Note that this situation is dramatically different from that in unfrustrated or weakly frustrated systems. There, also, many strong equivalence classes can exist; however, there are just two of them, related by spin-flip, that are distinguished by having minimal energy and a "minimal boundary condition." They correspond, at low temperatures, to two "bulk" Gibbs states (in unfrustrated systems these are the translation-invariant Gibbs states). In addition there may exist additional *Dobrushin* states,[25] which have higher energy and are naturally viewed as compositions of these "bulk" states separated by externally enforced interfaces.

What can be concluded in the highly frustrated spin glass? Clearly, we want to know whether the structure of strong equivalence classes implies an analog structure of Gibbs states at low temperatures. To answer this question, two points have to be addressed.

(i) Are there strong equivalence classes (in the thermodynamic limit)? Clearly we have seen that by choosing appropriate boundary conditions on a finite box Λ, we can obtain two ground states (for this finite volume Λ) whose symmetric difference is a surface spanning across Λ. The question to address is whether as $\Lambda \to \infty$ this surface remains sufficiently rigid so that the limiting infinite-volume ground states have a symmetric difference that intersects with finite probability finite boxes localized around the origin.

(ii) Are there infinite energy barriers between different equivalence classes, i.e, is there also a rich structure of *weak equivalence classes*? An affirmative answer to this question does not prove, but strongly hints, at the existence of many Gibbs states, while a negative one does exclude this possibility.

Let us first address question (i). Clearly, considering two states differing by a global spin-flip, their symmetric difference encloses all of Λ and thus also all finite regions Ω. Therefore, from this point of view, the possibility of having two disjoint Gibbs state differing by a global spin-flip exists in all dimensions.

Consider now two boundary conditions that differ by some loop λ on $\partial\Lambda$. The symmetric difference $[(\Gamma^1 \Delta \Gamma^2)_\infty]_\Lambda$ of the corresponding two ground states is a surface bounded by λ, such that $E_\Lambda(\Gamma^1 \Delta [(\Gamma^1 \Delta \Gamma^2)_\infty]_\Lambda)$ is minimal. If such a surface is typically *rigid*, i.e., is roughly a minimal surface of λ, we may force it to pass through a prescribed domain Ω by appropriately choosing λ. Therefore, states with different λ are distinguishable by local observables. If, on the contrary, this surface is "*rough*," its position in the interior of the system becomes undetermined as Λ grows, and we cannot expect to obtain disjoint Gibbs states in this way.

Which of the two possibilities is realized will depend on the dimensionality, and from general experience we expect increasing rigidity as the dimensionality grows. Which dimension is the critical one is not completely obvious. From experience with interfaces in dilute ferromagnets (a problem that is somewhat similar to the present question, but in itself highly nontrivial), we expect rigidity to occur in four dimensions,[29] while in three dimensions we still expect divergent fluctuations. If this analogy holds, this would imply that in the three-dimensional spin glass there are at most two disjoint Gibbs states, corresponding to the breaking of the global spin-flip symmetry. The existence of many ground states would only show up in dynamical effects. Due to extremely large relaxation times associated with the fluctuations of such interfaces, metastability, hysteresis, and pseudo-nonergodicity should be observed. In fact, it might be hard to distinguish this situation experimentally from one where actually infinitely many Gibbs states exist.[15]

In higher dimensions, with interfaces becoming rigid, we should expect infinitely many Gibbs states. As argued in Sec. 4, the quantity $\bar{\Delta}$ should provide a reliable measure for energy barriers, and weak equivalence classes should indeed correspond to low-temperature Gibbs states. If $[\Gamma^1 \Delta \Gamma^2]_\infty$ has dimension bigger than one, the energy barrier between Γ^1 and Γ^2, so measured, will be infinite, and Γ^1 and Γ^2 correspond to disjoint Gibbs states at low temperature. Due to the presence of the infinite frustration network, rather bizarre objects may arise as symmetric differences between ground states. In particular, objects with noninteger Hausdorff dimension, between 1 and d, are conceivable. These correspond to a spectrum of values for $\bar{\Delta}(\Gamma^1, \Gamma^2)$. It is tempting to conjecture that this will result in a (continuous) sequence of phase transitions below some critical T_c.

This may not be of great practical importance, since actual spin glasses do not exist in more than three dimensions. However, it makes contact with recent findings in the mean-field model, where, as now seems generally accepted, infinitely many low-temperature Gibbs state, arranged in a generation tree, have been discovered. This is in accordance with our argument that a spectrum of different, positive values of the energy barrier $\bar{\Delta}$ will appear in sufficiently large dimensions. Since $\bar{\Delta}$ is an ultrametric, the structure of Gibbs states emerging from our analysis appears to be closely related to that found in the Sherrington–Kirkpatrick model. Our considerations do, however, also indicate that the situation especially in three dimensions is delicate, and that mean-field results may not be adequate to describe it.

[15] Such questions have recently been discussed in the random-field Ising model (Ref. 30).

7. PERCOLATION THRESHOLDS IN THE SPIN GLASS PROBLEM, CONCLUSIONS

In this section we briefly recapitulate once more the different percolation phenomena encountered in our analysis of the spin glass problem and relate them to different features of the physics of spin glasses at low temperatures. This also provides us with a summary of some of the basic points of view adopted in this paper to construct our picture of spin glasses at low temperatures.

In Sec. 3 we have proven that, in *three or more* dimensions and for x sufficiently close to 1/2, a unique, infinite, connected network, Φ_∞, of frustration is present (almost surely). A preliminary computer study indicates that this is the case for

$$x_b < x < 1 - x_b$$

where $x_b \approx 0.09$, and that the density, ρ_∞, of Φ_∞ behaves like

$$\rho_\infty \sim (x - x_b)^{\beta_F}$$

with $\beta_F \approx 0.3$. The value $x_b \equiv x_c(1)$ of x at which Φ_∞ disappears is a new threshold of bond percolation in dimension $d \geqslant 3$. For reasons apparent from our discussion in Sec. 4–6 and below it would be most desirable to carry out a more detailed analysis—analytic and numerical—of the percolation of frustration.

The existence of an infinite frustration network, Φ_∞, for x sufficiently close to 1/2 has some important consequences for the spin glass problem:

(i) It proves that every configuration of spins in a glass has one infinite star-connected, gauge-invariant[16] contour, Γ_∞, such that

$$\partial \Gamma_\infty = \Phi_\infty \tag{7.1}$$

But even an Ising ferromagnet may exhibit an infinitely extended, connected contour in every spin configuration contributing to an extremal Gibbs or ground state *if appropriate boundary conditions are imposed.* Such states are the so-called Dobrushin states. They are inhomogeneous and break translation invariance. Physically, they describe the coexistence of a domain where the spins are primarily up with a domain where the spins are primarily down. In contrast, configurations in homogeneous states at low temperatures exhibit only a dilute gas of finite, closed contours whose probability is exponentially small in their length. Dobrushin states are stable against thermal fluctuations above two dimensions and are expected to survive up to $T = T_c$ (Ising) in four or more dimensions.

[16] Provided the external magnetic field vanishes as we always assume.

The existence of an infinite frustration network and of an infinite star-connected contour in every configuration of spins shows that in a spin glass, with x close to 1/2, there is no distinction between inhomogeneous (Dobrushin) states and homogeneous states. The infinite contour is always space-filling, and its Hausdorff dimension is the dimension, d, of the lattice, even at zero temperature.

(ii) The existence of Φ_∞ and Γ_∞ leads to the purely geometrical, or topological, notion of *strong equivalence classes of ground or Gibbs states*; see Sec. 4. Two states μ and μ' are strongly equivalent iff, for every $\Gamma \in \operatorname{supp} \mu$ and every $\Gamma' \in \operatorname{supp} \mu'$, $[\Gamma \Delta \Gamma']_\infty$ is a union of finite closed surfaces. By considering sequences of finite boxes, $(\Lambda_n)_{n=0}^\infty$, increasing to $\mathscr{Z}^d$ and imposing suitable boundary conditions on $\partial\Lambda_n$, and by joining up elements of $\Phi_\infty \cap \partial\Lambda_n$ by $(d-2)$-dimensional surfaces contained in $\partial\Lambda_n$, for each n, one can construct states μ and μ' such that $([\Gamma \Delta \Gamma']_\infty) \cap \Lambda_n$ cannot be a union of finite closed surfaces properly contained in Λ_n, for all $\Gamma \in \operatorname{supp} \mu$, $\Gamma' \in \operatorname{supp} \mu'$, and all n. However, the probability that $([\Gamma \Delta \Gamma']_\infty) \cap \Lambda_n$ intersects an arbitrary, given finite box Λ may tend to zero, as $n \to \infty$, because of divergent fluctuations of the coarse-grained position of $[\Gamma \Delta \Gamma']_\infty$. The problem of constructing strongly inequivalent states, μ and μ', is thus tied to the fluctuations of surfaces of the form $[\Gamma \Delta \Gamma']_\infty$. Inequivalent states will exist only if the fluctuations of the coarse-grained position of $[\Gamma \Delta \Gamma']_\infty$ remain bounded with probability one. By analogy with dilute Ising ferromagnets we expect that fluctuations remain bounded if the dimension of $[\Gamma \Delta \Gamma']_\infty$ is larger than two. This can be arranged in dimensions $d > 4$. (Thermal fluctuations of steps on $[\Gamma \Delta \Gamma']_\infty$ may delocalize $[\Gamma \Delta \Gamma']_\infty$ in dimensions $d \leqslant 4$.) Whether a spin glass phase with infinitely many inequivalent Gibbs states really exists at small, but finite, temperatures in $d \geqslant 4$ is tied to the question whether the rate of divergence of energy barriers, $\Delta((\Gamma, \sigma), (\Gamma', \sigma'))$ (see Sec. 4) is positive or not. If global spin-flip symmetry is broken, i.e., the Edwards–Anderson order parameter

$$q_{EA} = \lim_{\Lambda \to Z^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \sigma_x \rangle_\mu^2 \tag{7.2}$$

is positive, then Δ is expected to be equivalent to the more geometrical quantity $\bar{\Delta}$ (see Sec. 4), as we have argued in Sec. 4. Moreover, we expect $\bar{\Delta}(\Gamma, \Gamma')$ to be positive if the (Hausdorff) dimension of $([\Gamma \Delta \Gamma']_\infty)$ is larger than two.

We note that a distance between μ and μ' can be defined as follows:

$$\bar{\Delta}(\mu, \mu') \equiv \iint \bar{\Delta}(\Gamma, \Gamma')\, d\mu(\Gamma)\, d\mu'(\Gamma') \tag{7.3}$$

But we must expect that

$$\bar{\Delta}(\mu, \mu') = \bar{\Delta}(\Gamma, \Gamma')$$

for almost all $\Gamma \in \operatorname{supp} \mu$, $\Gamma' \in \operatorname{supp} \mu'$!

In conclusion, we expect that, in dimension $d \geqslant 4$ and for $T > 0$ sufficiently small, there exists an infinity of Gibbs states at a positive distance $\bar{\Delta}$ from each other, provided $q_{EA} > 0$, for all these states. The metric defined by $\bar{\Delta}$ turned out to be an *ultrametric.*

The positivity of $q_{EA}(\mu)$ is connected to another percolation process that we recall next. Let (Γ, σ) be some ground state. Consider a configuration (Γ', σ') and define

$$W(\Gamma, \Gamma') \equiv \Gamma \Delta \Gamma' \tag{7.4}$$

Clearly $\partial W(\Gamma, \Gamma') = \varnothing$. In order to construct (Γ', σ') from (Γ, σ), all spins inside $W(\Gamma, \Gamma')$ must be flipped. The energy difference between (Γ, σ) and (Γ', σ') is given by

$$\varepsilon(W(\Gamma, \Gamma')) \equiv E(\Gamma') - E(\Gamma) = \sum_{\langle ij \rangle^* \in W \cap \Gamma'} |J_{ij}| - \sum_{\langle ij \rangle^* \in W \cap \Gamma} |J_{ij}| \tag{7.5}$$

which is nonnegative by the definition of ground states. We call $\varepsilon(W(\Gamma, \Gamma'))$ the energy of the wall excitation $W(\Gamma, \Gamma')$. Generally $\varepsilon(W)$ may grow more slowly than the surface area $|W|$ of W. If typically $\varepsilon(W)/|W| \to 0$, as $|W| \to \infty$, then wall excitations will percolate, because the entropy of a wall W is proportional to $|W|$. We define $x_c(2)$ to be the percolation threshold for wall excitations. More precisely, we let $x_c(2)$ be the value of x such that, for $x_c(2) < x < 1 - x_c(2)$, there exist, for any dominant ground state $\Gamma \in \mathcal{G}$ and with probability one, connected wall excitations W enclosing arbitrarily large cubes centered at the origin such that $\varepsilon(W)/|W| \to 0$, as $|W| \to \infty$. (It is reasonable to replace the condition "$\varepsilon(W)/|W| \to 0$" by "$\varepsilon(W)$ remains uniformly bounded," as $|W| \to \infty$.)

Clearly, for these values of x, we expect the global spin-flip symmetry to remain unbroken, i.e., $q_{EA}(\mu) = 0$, for any Gibbs state μ (at least at positive temperature). The concept of percolation of finite-energy wall excitations easily extends to the positive temperature formalism.

In Sec. 6 we have presented what we believe to be a convinving case for the conjecture that, in two dimensions,

$$x_c(2) < \frac{1}{2} \tag{7.6}$$

We have also argued, less convincingly, that in dimension $d \geqslant 3$, $x_c(2)$ does not exist, i.e., wall excitations of finite energy above a ground state energy never percolate for $d \geqslant 3$, and hence one expects that, at low enough temperature, there exist extremal Gibbs states, μ, with $q_{EA}(\mu) > 0$. (We return to this conjecture below.)

It might be worthwhile to study the percolation of wall excitations of bounded energy within a simple model of Bernoulli percolation of $(d-1)$-cells of density x. Let W be a closed $(d-1)$-dimensional surface in $\mathscr{Z}^d$. Given a configuration Γ of occupied $(d-1)$-cells of a Bernoulli process, we define

$$\varepsilon(W) \equiv \| W \cap \Gamma| - |W \backslash W \cap \Gamma\| \tag{7.7}$$

The questions are whether

—there exist infinitely extended walls W with finite $\varepsilon(W)$;

—there exist walls W enclosing arbitrarily large cubes centered at the origin such that $\varepsilon(W)$ remains uniformly bounded.

In Sec. 5 we have argued that the correlation lenght $\xi(\beta)$ diverges for spin glasses in three or more dimensions, if x is sufficiently close to 1/2 and T is small enough. This was derived from

—global spin-flip symmetry breaking;

—the existence of wall excitations, $W(\Gamma, \Gamma')$, enclosing the sites 0 and y such that

$$\frac{\varepsilon(W(\Gamma, \Gamma'))}{|y|} \to 0, \qquad \text{as} \quad |y| \to \infty \tag{7.8}$$

for dominant ground states $\Gamma \in \mathscr{G}$, with probability decreasing to zero *less* than *exponentially rapidly*, as $|y| \to \infty$. Let $\mathscr{W}_{0y}$ be the event that, for ground states Γ, there exist connected walls, $W(\Gamma, \Gamma')$, enclosing 0 and y such that $\varepsilon(W(\Gamma, \Gamma')) \leqslant \text{const} |y|^{\alpha}$, with $\alpha < 1$. Let $x_c(3)$ be such that, for $x_c(3) < x < 1 - x_c(3)$,

$$\lim_{|y| \to \infty} \frac{1}{|y|} \ln \text{Prob}(\mathscr{W}_{0y}) = 0 \tag{7.9}$$

In Sec. 5 we have presented a calculation which strongly suggests that $x_c(3) < 1/2$, in arbitrary dimensions $d \geqslant 2$.

Hence, to conclude that the correlation length of an Ising spin glass diverges at low enough temperature in three or more dimensions, it suffices to show that the global spin-flip symmetry is broken for some $x > x_c(3)$. If $x_c(3)$ is well below 1/2, the calculations in Sec. 6 make it rather plausible that $x_c(2)$ is strictly larger than $x_c(3)$, [even if $x_c(2)$ may turn out to be slightly smaller than 1/2].[17] More careful heuristic estimates of these two threshold would thus be highly desirable!

Properties of wall excitations, the behavior of prob($\mathscr{W}_{0y}$), as $|y| \to \infty$, and the idea that in three or more dimensions $x_c(3) < x_c(2)$ can all be tested for the "simple" Bernoulli process of $(d-1)$-cells as described above. [See, in particular, (7.7)!] This points to some fascinating problems in ordinary d-dimensional bond percolation [dual to the percolation of $(d-1)$-cells] which may actually be solvable, at least with the help of a computer.

We should recall that our entire analysis of low-temperature Ising spin glasses has been carried out in zero magnetic field. It would be interesting to study the effect of external magnetic fields, h. One might expect to see sequences of transitions as T and h are varied.

In conclusion, we hope that, although we have *not* solved the spin glass problem, we have at least succeeded in posing it in a clearer way, in providing some hints of what the solution may look like and in exploring some possibilities that may make a solution more accessible. We also feel that we have isolated some *partial aspects* of the spin glass problem and some novel questions in *bond percolation theory* which may be more accessible to presently available analytical and numerical methods than the full-fledged spin glass problem. Some of those are presently being investigated.

ACKNOWLEDGMENTS

One of us (J.F.) thanks M. Aizenman, J. T. Chayes, L. Chayes, and L. Russo for introducing him to percolation theory and an enjoyable collaboration on three-dimensional bond and plaquette percolation which, along with often frustrating meditations on frustration, triggered his interest in the percolation of frustration and of "low-energy domain walls." We are deeply indebted to Michael Aizenman for sharing with us his insights into the percolation of frustration, correcting some misconceptions of one of us (J.F.), and contributing some technical arguments. J. Bricmont, J. T. Chayes, and L. Chayes have made useful comments on our thoughts and provided encouragement. We are greatful to G. Baskaran for informing us about his independent attempts of using percolation-type arguments in analyzing short-range spin glasses before publication, and to E. Brézin and T. Spencer for their stimulating interest in our work. Finally, one of us (J.F.) thanks the Institute for Advanced Study, Princeton, for hospitality during a period when some of our thoughts were developed, and the U.S.N.S.F. for partial financial support under grant DMS-8405264.

[17] We expect $x_c(3)$ to be at most about 0.23 in three dimensions; see Sec. 3.

REFERENCES

1. P. W. Anderson, Survey of theories of spin glasses, in *Amorphous Magnetism II*, R. A. Levy and R. Hasegawa, eds. (Plenum Press, New York, 1977).
2. G. Toulouse, Frustration and disorder in statistical mechanics: spin glasses in historical perspective, in *Heidelberg Colloquium on Spin Glasses*, J. L. van Hemmen and I. Morgenstern, eds. (Springer Verlag, Berlin, Heidelberg, New York, Tokyo 1983), and other papers in this volume.
3. K. H. Fischer, *Phys. Status Solidi* (*b*) **116**:357 (1983).
4. R. N. Bhatt and A. P. Young, *Phys. Rev. Lett.* **54**:340 (1985); A. T. Ogielski and I. Morgenstern, *Phys. Rev. Lett.* **54**:428 (1985).
5. N. Sourlas, *J. Phys. Lett.* **45**:L969 (1984).
6. S. Kirkpatrick and P. Sherrington, *Phys. Rev.* **B17**:4384 (1978).
7. G. Parisi, *J. Phys.* **A13**:1101 (1980); **A13**:1887 (1980) *Phys. Rev. Lett.* **50**:1946 (1983).
8. M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, *Phys. Rev. Lett.* **52**:1156 (1984); *J. Phys.* **45**:843 (1984).
9. V. Cannella and J. A. Mydosh, *Phys. Rev.* **B6**:4220 (1972).
10. P. W. Anderson, Princeton University preprint, 1985, and references therein.
11. S. F. Edwards and P. W. Anderson, *J. Phys.* **F5**:1965 (1975).
12. G. Toulouse, *Commun. Phys.* **2**:115 (1977).
13. J. Fröhlich, Mathematical Aspects of the Physics of Disordered Systems, in *Proc. of the 1984 Les Houches Summer School*, K. Osterwalder and R. Stora, eds. (Plenum Press, New York), 1986, to appear.
14. J. E. Avron, G. Röpstorff, and L. S. Schulman, *J. Stat. Phys.* **26**:25 (1981).
15. D. C. Mattis, *Phys. Lett.* **A50**:421 (1976).
16. H. E. Kesten, *Percolation Theory for Mathematicians*, (Birkhäuser Verlag, Boston, Basel, Stuttgart, 1983); J. W. Essam, Percolation and cluster size, in *Phase Transitions and Critical Phenomena*, Vol. 2, C. Domb and M. S. Green, eds. (Academic Press, London, 1972).
17. M. Aizenman and E. Lieb, *J. Stat. Phys.* **24**:279 (1981).
18. S. A. Pirogov and Ya. G. Sinai, *Teor. Mat. Fiz.* **25**:358 (1975) [*Theor. Math. Phys.* **25**: 1185 (1976); *Teor. Mat. Fiz.* **26**:61 (1976) [*Theor. Math. Phys.* **26**:39 (1976)].
19. J. Slawny, *J. Stat. Phys.* **20**:711 (1979); M. Zahradnik, *Commun. Math. Phys.* **93**:559 (1984).
20. M. E. Fisher, *J. Appl. Phys.* **38**:981 (1967); N. D. Mermin, *J. Phys. Soc. Jpn. Suppl.* **26**:263 (1969).
21. R. Peierls, *Proc. Cambridge Philos. Soc.* **32**:477 (1936); R. Griffiths, *Phys. Rev.* **136**:A437 (1964).
22. F. J. Dyson, *Commun. Math. Phys.* **12**:91, 212 (1969).
23. J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **81**:527 (1981).
24. Ch.-E. Pfister, *Commun. Math. Phys.* **79**:181 (1981).
25. R. L. Dobrushin, *Theor. Prob. Appl.* **73**:582 (1972).
26. M. Aizenman, *Commun. Math. Phys.* **73**:83 (1980).
27. J. Vannimenus and G. Toulouse, *J. Phys.* **C10**:L537 (1977); S. Kirkpatrick, *Phys. Rev.* **B16**:4630 (1977); I. Morgenstern and K. Binder, *Phys. Rev.* **B22**:288 (1980).
28. B. Derrida, *Phys. Rev.* **B24**:2613 (1981).
29. D. A. Huse and Ch. L. Henley, *Phys. Rev. Lett.* **54**:2704 (1985); M. Kardar, *Phys. Rev. Lett.* **55**:2923 (1985); A. Bovier, J. Fröhlich, and U. Glaus, *Phys. Rev. B* (1986), to appear.
30. J. Villain, *Phys. Rev. Lett.* **52**:1543 (1984); in *Scaling Phenomena in Disordered Systems*, R. Rhynn, ed., (1986, to appear); R. Bruinsma and G. Aeppli, *Phys. Rev. Lett.* **52**:1547 (1984).